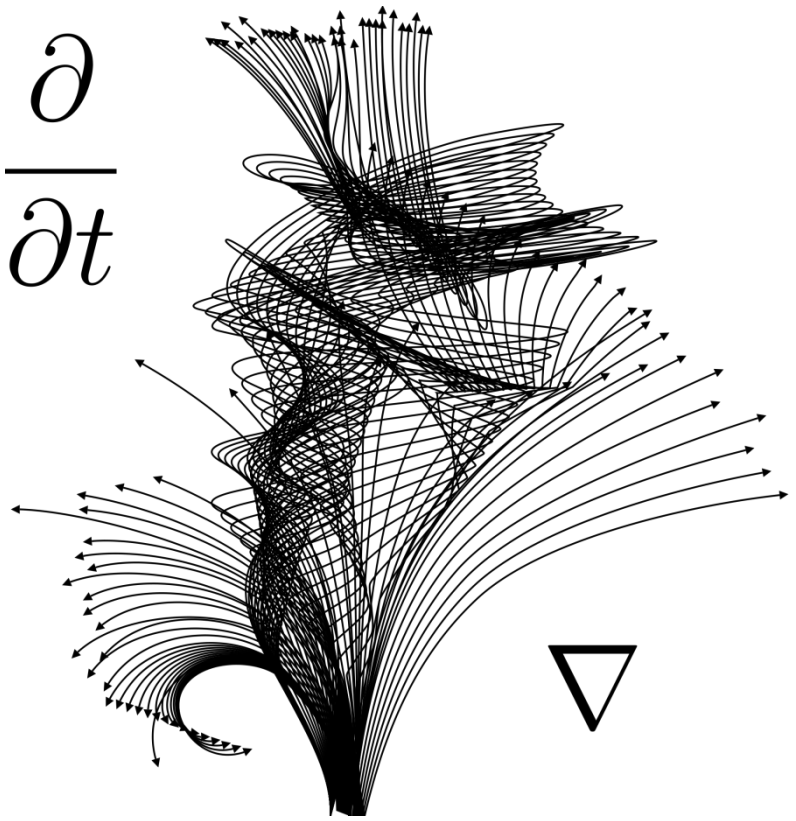
Partial Differential Equations

Department of Mathematics and Computer Applications



Fourth Year

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LESSON 1: Introduction to Partial Differential Equations

Definition: A **partial differential equation** is an equation that contains partial derivatives. In contrast to ordinary differential equations (O.D.E.s), where the unknown function depends only on *one variable*, in P.D.E.s, the unknown function depends on several variables (like temperature $u(x,t)$ depends both on location *x* and time *t*).

Note that for notational simplicity we have called

$$u_t = \frac{\partial u}{\partial t},\ u_x = \frac{\partial u}{\partial x},\ u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

A Few Well-Known PDEs

- $u_t = u_{xx}$ (Heat equation in one dimension).
- $u_t = u_{xx} + u_{yy}$ (Heat equation in two dimensions).
- $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (Laplace's equation in polar coordinates).
- $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ (Wave equation in three dimensions).
- $u_{tt} = u_{xx} + \alpha u_t + \beta u$ (Telegraph equation).

Note:

The unknown function *u* always depends on *more* than **one** variable. The variable *u* (which we differentiate) is called the **dependent variable**, whereas the ones we differentiate *with respect to* are called the **independent variables**. For example, it is clear from the equation

$$u_t = u_{xx}$$

that the dependent variable $u(x,t)$ is a function of two independent variables *x* and *t*, whereas in the equation

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$u(r,\theta,t)$ depends on *r*, θ, and *t*.

Why Are P.D.E.s Useful?

Most of the natural laws of physics, such as Maxwell's equations, Newton's law of cooling, the Navier-Stokes equations, Newton's equations of motion, and Schrodinger's equation of quantum mechanics, are stated (or can be) in terms of P.D.E.s, that is, these laws describe physical phenomena by relating space and time derivatives.

Derivatives occur in these equations because the derivatives represent *natural things* (like velocity, acceleration, force, friction, flux, current).

Hence, we have equations relating partial derivatives of some unknown quantity that we would like to find.

How Do You Solve a Partial Differential Equation?

The most important methods are those that change PDEs into ODEs. Ten useful techniques are:

1. ***Separation of Variables.***
   This technique reduces a PDE in *n* variables to *n* ODEs.
2. ***Integral Transforms.***
   This procedure reduces a P.D.E. in *n* independent variables to one in $n-1$ variables; hence, a PDE in two variables could be changed to an ODE.
3. ***Change of Coordinates.***
   This method changes the original PDE to an ODE or else another PDE (an easier one) by changing the coordinates of the problem.
4. ***Transformation of the Dependent Variable.***
   This method transforms the unknown of a PDE into a new unknown that is easier to find.
5. ***Numerical Methods.***
   These methods change a PDE to a system of *difference equations* that can be solved by means of iterative techniques on a computer; in many cases, this is the only technique that will work.
6. ***Pertubation Methods.***
   This method changes a nonlinear problem into a sequence of *linear ones* that approximates the nonlinear one.
7. ***Impulse-response Technique.***
   This procedure decomposes initial and boundary conditions of the problem into *simple impulses* and finds the response to each impulse. The overall response is then found by adding these simple responses.
8. ***Integral Equations.***
   This technique changes a PDE to an integral equation (an equation where the unknown is inside the integral). The integral equation is then solved by various techniques.
9. ***Calculus of Variations Methods.***
   These methods find the solution to P.D.E.s by reformulating the equation as a *minimization problem.* It turns out that the minimum of a certain expression (very likely the expression will stand for total energy) is also the solution to the (P.D.E.).
10. ***Eigenfunction Expansion.***
    This method attempts to find the solution of a P.D.E. as an infinite sum of *eigenfunctions.* These *eigenfunctions* are found by solving what is known as an eigenvalue problem corresponding to the original problem.

Kinds of P.D.E.s

Partial differential equations are classified according to many things. Classification is an important concept because the general theory and methods of solution usually apply only to a given class of equations. Six basic classifications are:

1. ***Order of the PDE.***

   The order of a PDE is the order of the *highest partial derivative* in the equation, for example,

   $u_t = u_{xx}$ (Second order)
   $u_t = u_x$ (First order)
   $u_t = uu_{xxx} + \sin x$ (third order)

2. ***Number of Variables.***

   The number of variables is the *number of independent* variables, for example,

   $u_t = u_{xx}$ (Two variables: *x* and *t*).
   $u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ (Three variables: *r, θ,* and *t*).

3. ***Linearity.***

   Partial differential equations are either *linear* or *nonlinear*. In the linear ones, the dependent variable *u* and all its derivatives appear in a linear fashion (they are not multiplied together or squared, for example).

More precisely, a **second-order linear equation in two variables** is an equation of the form

$$\boldsymbol{A}u_{xx} + \boldsymbol{B}u_{xy} + \boldsymbol{C}u_{yy} + \boldsymbol{D}u_x + \boldsymbol{E}u_y + \boldsymbol{F}u = \boldsymbol{G} \qquad \textbf{(1)}$$

where ***A***, ***B***, ***C***, ***D***, ***E***, ***F***, and ***G*** can be *constants* or given *functions* of *x* and *y;* for example,

$u_{tt} = e^{-t}u_x + \sin t$ (linear)
$uu_{xx} + u_t = 0$ (nonlinear)
$u_{xx} + yu_{yy} = 0$ (linear)
$xu_x + yu_y + u^2 = 0$ (nonlinear)

4. ***Homogeneity.***

   The equation (1) is called **homogeneous** if the right-hand side *G(x,y)* is identically zero for all *x* and *y*. If *G(x,y)* is not identically zero, then the equation is called **nonhomogeneous.**

5. ***Kinds of Coefficients.***

   If the coefficients *A, B, C, D, E,* and Fin equation (1) are constants, then (1) is said to have **constant coefficients** (otherwise, variable coefficients).

6. Three Basic Types of Linear Equations. All linear PDEs like equation (1) are either
   (a) Parabolic.
   (b) Hyperbolic.
   (c) Elliptic.

Parabolic equations

Describe heat flow and diffusion processes and satisfy the property $B^2 - 4AC = 0$.

Hyperbolic equations

Describe vibrating systems and wave motion and satisfy the property $B^2 - 4AC > 0$.

Elliptic equations

Describe steady-state phenomena and satisfy the property $B^2 - 4AC < 0$.

Examples:

1. $u_t = u_{xx}$ $B^2 - 4AC = 0$ (Parabolic).
2. $u_{tt} = u_{xx}$ $B^2 - 4AC = 4$ (Hyperbolic).
3. $u_{\xi\eta} = 0$ $B^2 - 4AC = 1$ (Hyperbolic).
4. $u_{xx} + u_{yy} = 0$ $B^2 - 4AC = -4$ (Elliptic).
5. $yu_{xx} + u_{yy} = 0$ $B^2 - 4AC = -4y$ $\begin{cases}\text{Elliptic } y > 0\\ \text{Parabolic } y = 0\\ \text{Hyperbolic } y < 0\end{cases}$

<table>
<tr><td>Linearity</td><td colspan="4">Linear</td><td colspan="3">Nonlinear</td></tr>
<tr><td>Order</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td></td><td>m</td></tr>
<tr><td>Kinds of coefficients (linear equations)</td><td colspan="4">Constant</td><td colspan="3">Variable</td></tr>
<tr><td>Homogeneity (linear equations)</td><td colspan="4">Homogeneous</td><td colspan="3">Nonhomogeneous</td></tr>
<tr><td>Number of variables</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td></td><td>n</td></tr>
<tr><td>Basic type (linear equations)</td><td colspan="3">Hyperbolic</td><td colspan="2">Parabolic</td><td colspan="2">Elliptic</td></tr>
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Classification diagram for partial differential equations

PROBLEMS

1. Classify the following equations according to all the properties we've discussed in diagram:
   a) $u_t = u_{xx} + 2u_x + u.$
   b) $u_t = u_{xx} + e^{-t}.$
   c) $u_{xx} + 3u_{xy} + u_{yy} = sin\, x.$
   d) $u_{tt} = u\, u_{xxxx} + e^{-t}.$
2. How many solutions to the PDE $u_t = u_{xx}$ can you find? Try solutions of the form $u(x,t) = e^{ax+bt}$.
3. If $u_1(x,t)$ and $u_2(x,t)$ satisfy equation (1), then is it true that the sum satisfies it? if yes, prove it.

***LESSON* 2: Diffusion-Type Problems (Parabolic Equations)**

The Mathematical Model of the Heat-Flow Experiment

The description of our physical problem requires three types of equations

1. The P.D.E. describing the physical phenomenon of heat flow.
2. The *boundary conditions* describing the physical nature of our problem on the boundaries.
3. The *initial conditions* describing the physical phenomenon at the start of the experiment.

The Heat Equation

The basic equation of one-dimensional heat flow is the relationship

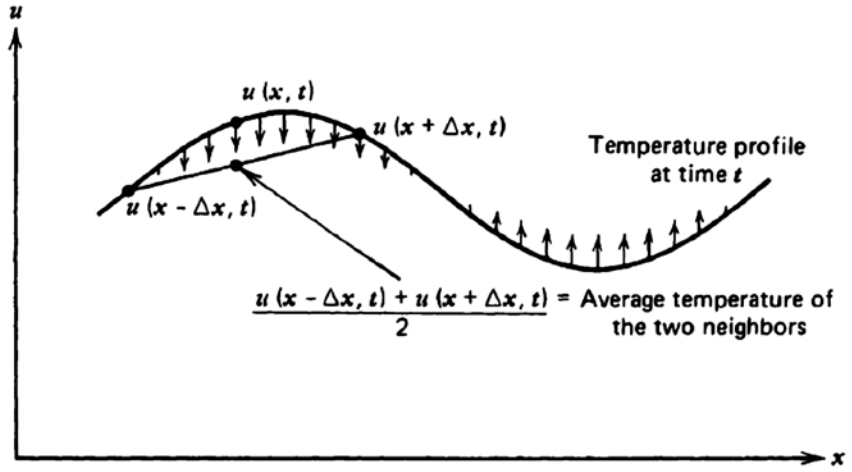
P.D.E. $u_t = \alpha^2 u_{xx} \qquad 0 < x < L \qquad 0 < t < \infty$

which relates the quantities

u_t= the *rate of change* in temperature with respect to time (measured in deg/sec) and

u_{xx}= the *concavity* of the temperature profile $u(x,t)$ (which essentially compares the temperature at one point to the temperature at neighboring points).

This equation will be derived from the basic *conservation of heat equation* in later lessons, but for the time being, we examine it by itself. This equation simply says that the temperature $u(x,t)$ (at some point along the rod *x* and at some point in time *t)* is increasing ($u_t > 0$) or decreasing ($u_t < 0$) according to whether u_{xx} is positive or negative. The Figure illustrates the change in temperature at different points along the rod.



Arrows indicating change in temperature according to $u_t = \alpha^2 u_{xx}$

Boundary Conditions

All physical problems have boundaries of some kind, so we must describe mathematically what goes on there in order to adequately describe the problem. In our experiment, the boundary conditions (BCs) are quite easy. Since the temperature u was fixed for all time $t > 0$ at T_1 and T_2 at the two ends $x = 0$ and $x = L$, we would simply say

$$\text{BCs} \quad \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \qquad 0 < t < \infty$$

Initial Conditions

All physical problems must start from some value of time (generally called $t = 0$), so we must specify the physical apparatus at this time. Since we started monitoring the rod temperature in our example from the time the rod had achieved a constant temperature of T_0, we have

$$\text{I.C.} \quad u(x,0) = T_0 \qquad 0 \le x \le L$$

We have now mathematically described the experiment. By writing equations together, we have what is called an *initial-boundary-value problem* (I.B.V.P.).

$$\begin{aligned} &\text{P.D.E.} && u_t = \alpha^2 u_{xx} && 0 < x < L && 0 < t < \infty \\ &\text{BCs} && \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} && 0 < t < \infty \\ &\text{I.C.} && u(x,0) = T_0 && 0 \le x \le L \end{aligned} \qquad \textbf{(2)}$$

The interesting thing here, which is not at all obvious, is that there is only one function $u(x,t)$ that satisfies the problem (2), and that function will describe the temperature of the rod. Hence, our goal in the near future will be to find that unique solution $u(x,t)$ to (2).

***LESSON* 3: Separation of Variables**

Separation of variables is one of the oldest techniques for solving initial boundary-value problems (IBVPs) and applies to problems where

1. The PDE is linear and homogeneous (not necessarily constant coefficients).
2. The boundary conditions are of the form

$$\alpha u_x(0,t) + \beta u(0,t) = 0$$
$$\gamma u_x(1,t) + \delta u(1,t) = 0$$

where α, β, γ and δ are constants (boundary conditions of this form are called linear homogeneous BCs).

Instead of showing how the method works in general, let's apply it to a specific problem (later we will discuss it in more generality). Consider the IBVP (diffusion problem)

$$\begin{array}{lll} \text{P.D.E.} & u_t = \alpha^2 u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{B.C.s} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{I.C.} & u(x,0) = \phi(x) & 0 \le x \le 1 \end{array}$$

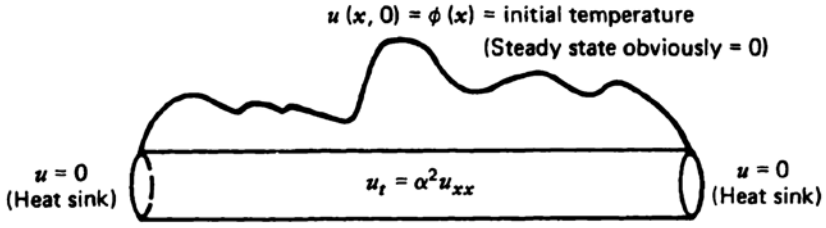


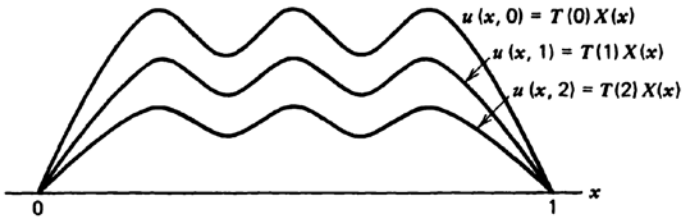
Diagram of the diffusion problem

Overview of Separation of Variables

Separation of variables looks for simple-type solutions to the PDE of the form

$$u(x,t) = X(x)T(t)$$

where $X(x)$ is some function of x and $T(t)$ is some function of t. The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic "shape" for different values of time t.



Graph of $X(x)T(t)$ for different values of t.

The general idea is that it is possible to find an infinite number of these solutions to the PDE (which, at the same time, also satisfy the BCs). These simple functions $u_n(x,t) = X_n(x)T_n(t)$ (called **fundamental solutions)** are the building blocks of our problem, and the solution $u(x,t)$ we are looking for is found by adding the simple fundamental solutions $X_n(x)T_n(t)$ in such a way that the resulting sum

$$\sum_{n=0}^{\infty} A_n X_n(x) T_n(t)$$

satisfies the initial conditions. In as much as this sum still satisfies the PDE and the BCs, we now have the solution to our problem.

Separation of Variables

***Step* 1:** (Finding elementary solutions to the PDE)

We wish to find the function $u(x,t)$ that satisfies the following four conditions:

$$\begin{array}{lll} \text{P.D.E.} & u_t = \alpha^2 u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{B.C.s} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{I.C.} & u(x,0) = \phi(x) & 0 \le x \le 1 \end{array} \qquad \textbf{(3)}$$

To begin, we look for solutions of the form $u(x,t) = X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE and solving for $X(x)T(t)$. Making this substitution gives

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Now, here is the part that makes all this work: If we divide each side of this equation by $\alpha^2 X(x)T(t)$, we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

and obtain what is called separated variables, that is, the left side of the equation depends only on t and the right side, only on x. Inasmuch as x and tare independent of each other, each side must be a fixed constant (say k); hence, we can write

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k$$

or

$$T' - k\alpha^2 T = 0$$
$$X'' - kX = 0$$

So now we can solve each of these two ODEs, multiply them together to get a solution to the PDE (note that we have essentially changed a second-order PDE to two ODEs).

However, we now make an important observation, namely, that we want the separation constant k to be ***negative*** (or else the $T(t)$ factor doesn't go to zero as $t \to \infty$). With this in mind, it is general practice to rename $k = -\lambda^2$, where λ is nonzero ($-\lambda^2$ is guaranteed to be negative). Calling our separation constant by its new name, we can now write the two ODEs as

$$T' + \lambda^2 \alpha^2 T = 0$$
$$X'' + \lambda^2 X = 0$$

We will now solve these equations. Both equations are standard-type ODEs and have solutions

$T(t) = Ae^{-\lambda^2 \alpha^2 t}$ (A an arbitrary constant)
$X(x) = A\,sin(\lambda x) + B\,cos(\lambda x)$ (A, B arbitrary)

and hence all functions

$$u(x,t) = e^{-\lambda^2 \alpha^2 t}[\boldsymbol{A}\,sin(\lambda x) + \boldsymbol{B}\,cos(\lambda x)] \quad \text{(with } \boldsymbol{A}, \boldsymbol{B} \text{ and } \lambda \text{ arbitrary)}$$

will satisfy the PDE $u_t = \alpha^2 u_{xx}$ this verification is problem 3 in the problem set. At this point, we have an infinite number of functions that satisfy the PDE.

***Step* 2:** (Finding solutions to the PDE and the BCs)

We are now to the point where we have many solutions to the PDE but not all of them satisfy the BCs or the IC. The next step is to choose a certain subset of our current crop of solutions

$$e^{-\lambda^2 \alpha^2 t}[\boldsymbol{A}\,sin(\lambda x) + \boldsymbol{B}\,cos(\lambda x)] \qquad \textbf{(4)}$$

that satisfy the boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

To do this, we substitute our solutions (4) into these BCs, getting

$$u(0,t) = \boldsymbol{B}e^{-\lambda^2 \alpha^2 t} = 0 \Longrightarrow \boldsymbol{B} = 0$$
$$u(1,t) = \boldsymbol{A}e^{-\lambda^2 \alpha^2 t}\,sin\,\lambda = 0 \Longrightarrow sin\,\lambda = 0$$

This last B.C. restricts the separation constant λ from being any nonzero number, it must be a root of the equation $sin\,\lambda = 0$. In other words, in order that $u(1,t) = 0$, it is necessary to *pick*

$$\lambda = \mp\pi, \mp 2\pi, \mp 3\pi, \ldots$$

or

$$\lambda = \mp n\pi, \quad n = 1,2,3, \ldots$$

Note that the last B.C. could also imply $A = 0$, but if we choose this, we would get the zero solution in (4).

We have now finished the second step, we have found an infinite number of functions

$$u_n(x,t) = \boldsymbol{A}_n e^{-(n\pi\alpha)^2 t}\,sin\,n\pi\,x, \qquad n = 1,2,3, \ldots \qquad \textbf{(5)}$$

Each one satisfying the PDE and the B.C.s. These are the building blocks of the problem, and our desired solution will be a certain sum of these simple functions; the specific sum will depend on the initial conditions.

***Step* 3:** (Finding the solution to the PDE, BCs, and the IC)

The last step (and probably the most interesting from a mathematical point of view) is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} \boldsymbol{A}_n e^{-(n\pi\alpha)^2 t} \sin n\pi x$$

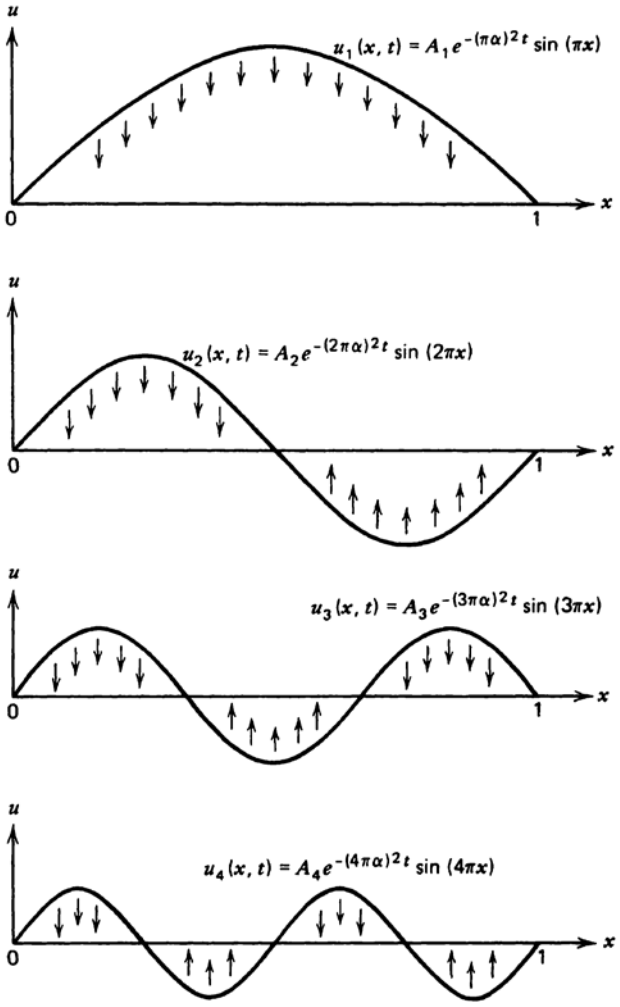
in such a way (pick the coefficients $\boldsymbol{A}_n$) that the initial condition

$$u(x,0) = \phi(x)$$

is satisfied. Substituting the sum into the I.C. gives

$$\phi(x) = \sum_{n=1}^{\infty} \boldsymbol{A}_n \sin n\pi x$$

Hence, how to find the coefficients A_n.



Fundamental solutions $u_n(x,t) = \boldsymbol{A}_n e^{-(n\pi\alpha)^2 t} \sin n\pi x$

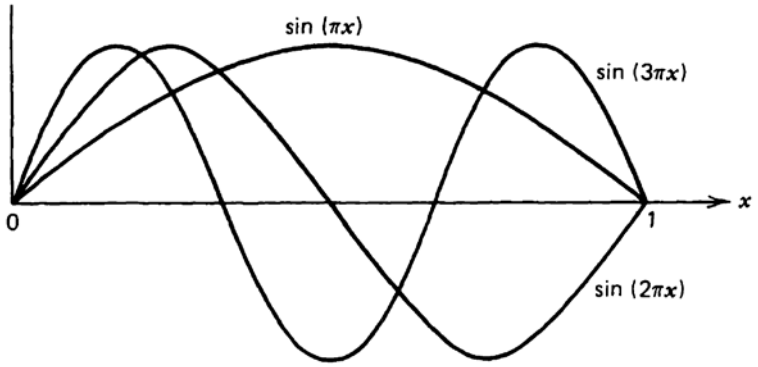
One uses a property of the functions

$$sin\, n\pi\, x, \qquad n = 1,2,3, \ldots$$

known as orthogonality. It turns out that these functions are orthogonal to each other in the sense

$$\int_0^1 sin(m\pi x)\, sin(n\pi x)\, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

This property can be illustrated by looking at the graphs of these functions



Orthogonal sequence of functions

So, we are now in position to solve for the coefficients in the expression

$$\phi(x) = \boldsymbol{A}_1\, sin\, \pi x + \boldsymbol{A}_2\, sin\, 2\pi x + \boldsymbol{A}_3\, sin\, 3\pi x + \boldsymbol{A}_4\, sin\, 4\pi x + \cdots$$

We multiply each side of this equation by $sin(m\pi x)$ (m, an arbitrary integer) and integrate from zero to one; doing this, we get

$$\int_0^1 \phi(x)\, sin(m\pi x)\, dx = \boldsymbol{A}_m \int_0^1 sin^2(m\pi x)\, dx = \frac{1}{2}\boldsymbol{A}_m$$

(all other terms drop out due to orthogonality). Solving for $\boldsymbol{A}_m$ gives

$$\boldsymbol{A}_m = 2\int_0^1 \phi(x)\, sin(m\pi x)\, dx$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \boldsymbol{A}_n e^{-(n\pi\alpha)^2 t}\, sin(n\pi x)$$

where the coefficients $\boldsymbol{A}_n$ are given by

$$\boldsymbol{A}_n = 2\int_0^1 \phi(x)\, sin(n\pi x)\, dx$$

We can check this answer to see that it satisfies all four of our original conditions in the problem.

Example: Solve the following heat heat-flow problem:

P.D.E. $u_t = \alpha^2 u_{xx}$ $0 < x < 1$ $0 < t < \infty$

B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$ $0 < t < \infty$

I.C. $u(x,0) = sin(\pi x) + \frac{1}{2} sin(3\pi x)$ $0 \le x \le 1$

By Separation of Variables.

Solution:

Let $u(x,t) = X(x)T(t)$

$u_t = X(x)T'(t)$

$u_{xx} = X''(x)T(t)$

$[XT' = \alpha^2 X''T] \div \alpha^2 XT$

$\frac{XT'}{\alpha^2 XT} = \frac{\alpha^2 X''T}{\alpha^2 XT}$

$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k,$ k is constant.

$T' - k\alpha^2 T = 0$

$X'' - kX = 0$

put $k = -\lambda^2$

$T' + \lambda^2 \alpha^2 T = 0$

$T(t) = Ae^{-\lambda^2 \alpha^2 t}$

$X'' + \lambda^2 X = 0$

$X(x) = A\, sin(\lambda x) + B\, cos(\lambda x)$

$u(x,t) = e^{-\lambda^2 \alpha^2 t}[\boldsymbol{A}\, sin(\lambda x) + \boldsymbol{B}\, cos(\lambda x)]$

Now, substituting the B.C.s $u(0,t) = 0$ $u(1,t) = 0$

$u(0,t) = e^{-\lambda^2 \alpha^2 t}[\boldsymbol{A}\, sin(\lambda 0) + \boldsymbol{B}\, cos(\lambda 0)] = \boldsymbol{B} e^{-\lambda^2 \alpha^2 t} = 0, \Longrightarrow \boldsymbol{B} = 0$

$u(1,t) = e^{-\lambda^2 \alpha^2 t} \boldsymbol{A}\, sin(\lambda) = 0, \boldsymbol{A} \neq 0 \Rightarrow sin(\lambda) = 0 \Longrightarrow \lambda = \pm n\pi$

Then

$u_n(x,t) = \boldsymbol{A}_n e^{-(\alpha n\pi)^2 t}\, sin(n\pi x)$

$$u(x,t) = \sum_{n=1}^{\infty} \boldsymbol{A}_n e^{-(\alpha n\pi)^2 t}\, sin\, n\pi\, x$$

Apply the I.C. $u(x,0) = sin(\pi x) + \frac{1}{2} sin(3\pi x)$

$$u(x,0) = sin(\pi x) + \frac{1}{2} sin(3\pi x) = \sum_{n=1}^{\infty} \boldsymbol{A}_n e^0\, sin\, n\pi\, x = \sum_{n=1}^{\infty} \boldsymbol{A}_n\, sin\, n\pi\, x$$

multiply both side by $sin(m\pi x)$ and integrate from zero to one, we get

$$\int_0^1 \left[sin(\pi x) + \frac{1}{2}sin(3\pi x)\right] sin(m\pi x)\, dx = \sum_{n=1}^{\infty} \boldsymbol{A}_m \int_0^1 sin(n\pi x)\, sin(m\pi x)\, dx$$

Form the orthogonality property of sin function, we get

$$\boldsymbol{A}_m = 2\int_0^1 \left[sin(\pi x) + \frac{1}{2}sin(3\pi x)\right] sin(m\pi x)\, dx$$

Then

$$\boldsymbol{A}_1 = 1$$
$$\boldsymbol{A}_2 = 0$$
$$\boldsymbol{A}_3 = \frac{1}{2}$$
$$\boldsymbol{A}_4 = \boldsymbol{A}_5 = \cdots = 0$$

The solution is

$$u(x,t) = e^{-(\pi\alpha)^2 t}\, sin(\pi x) + \frac{1}{2}e^{-(3\pi\alpha)^2 t}\, sin(3\pi x)$$

PROBLEMS

1- Show that $u(x,t) = e^{-\lambda^2\alpha^2 t}[A\, sin(\lambda x) + B\, cos(\lambda x)]$ satisfies the PDE $u_t = \alpha^2 u_{xx}$ for arbitrary *A*, *B* and λ.

2- Show $\int_0^1 sin(m\pi x)\, sin(n\pi x)\, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$

HINT Use the identity

$$sin(mx)\, sin(nx) = \frac{1}{2}[cos(m-n)x - cos(m+n)x]$$

3- what is the solution to the IBVP

P.D.E. $u_t = \alpha^2 u_{xx}$ $0 < x < 1$ $0 < t < \infty$

B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$ $0 < t < \infty$

I.C. $u(x,0) = 1$ $0 \leq x \leq 1$

4- What is the solution to problem 3 if the IC is changed to

$$u(x,0) = sin(2\pi x) + \frac{1}{3}sin(4\pi x) + \frac{1}{5}sin(6\pi x)$$

5- What is the solution to problem 3 if the IC is changed to

$$u(x,0) = x^2 - x, \qquad 0 \leq x \leq 1$$

***LESSON* 4: Transforming Nonhomogeneous BCs into Homogeneous Ones**

Consider heat flow in an insulated rod where the two ends are kept at constant temperatures k_1 and k_2; that is,

$$\begin{array}{lll} \text{P.D.E.} & u_t = \alpha^2 u_{xx} & 0 < x < L \quad 0 < t < \infty \\ \text{B.C.s} & \begin{cases} u(0,t) = k_1 \\ u(L,t) = k_2 \end{cases} & 0 < t < \infty \\ \text{I.C.} & u(x,0) = \phi(x) & 0 \le x \le L \end{array} \qquad \textbf{(6)}$$

The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables. However, it is obvious that the solution will have a steady-state solution (solution when $t = \infty$) that varies linearly (in x) between the boundary temperatures k_1 and k_2.

It seems reasonable to think of our temperature $u(x,t)$ as the sum of two parts

$$u(x,t) = \text{steady state} + \text{transient}$$

Our goal is to find the *transient* $U(x,t)$. By substituting

$$u(x,t) = \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

in the original problem (6), we will arrive at a new problem in $U(x,t)$. We can then solve this new one for $U(x,t)$ and add it to the steady state to get $u(x,t)$. Carrying out this simple substitution in (6) gives us

$$\begin{array}{lll} \text{P.D.E.} & U_t = \alpha^2 U_{xx} & 0 < x < L \quad 0 < t < \infty \\ \text{B.C.s} & \begin{cases} U(0,t) = 0 \\ U(L,t) = 0 \end{cases} & 0 < t < \infty \\ \text{I.C.} & U(x,0) = \phi(x) - \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] = \bar{\phi}(x) & \end{array}$$

This problem (fortunately) has a homogeneous PDE as well as homogeneous B.C.s, and so we can solve it by separation of variables; in fact, the reader probably remembers the solution:

$$U(x,t) = \sum_{i=1}^{\infty} \boldsymbol{a}_n e^{-(n\pi\alpha)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\boldsymbol{a}_n = \frac{2}{L}\int_0^L \bar{\phi}(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

So much for rods with *fixed* temperatures at the boundaries. What about more realistic-type derivative BCs with *time-varying* right-hand sides? The ideas are similar to the previous problem but a little more complicated.

Transforming Time Varying BCs to Zero BCs

Consider the typical problem

$$\begin{array}{lll} \text{P.D.E.} & u_t = \alpha^2 u_{xx} \quad 0 < x < L & 0 < t < \infty \\ \text{B.C.s} & \begin{cases} u(0,t) = g_1(t) \\ u_x(L,t) + hu(L,t) = g_2(t) \end{cases} & 0 < t < \infty \\ \text{I.C.} & u(x,0) = \phi(x) \quad 0 \le x \le L & \end{array} \qquad \textbf{(7)}$$

To change these nonzero BCs to homogeneous ones, we (after some trial and error) seek a solution of the form

$$u(x,t) = A(t)\left[1 - \frac{x}{L}\right] + B(t)\left[\frac{x}{L}\right] + U(x,t)$$

where $A(t)$ and $B(t)$ are chosen so that the steady-state part

$$S(x,t) = A(t)\left[1 - \frac{x}{L}\right] + B(t)\left[\frac{x}{L}\right]$$

satisfies the BCs of the problem. In this way, the transformed problem in $U(x,t)$ will have homogeneous BCs. Substituting $S(x,t)$ into the BCs

$$S(x,t) = g_1(t)$$
$$S_x(L,t) + hS(L,t) = g_2(t)$$

gives us two equations in which we can solve for $A(t)$ and $B(t)$. Doing this, we get

$$A(t) = g_1(t)$$
$$B(t) = \frac{g_1(t) + Lg_2(t)}{1 + Lh}$$
$$S_x(x,t) = \frac{-A(t) + B(t)}{L}$$
$$S_x(L,t) = \frac{-A(t) + B(t)}{L}$$
$$S(x,t) = B(t)$$
$$\frac{-A(t) + B(t)}{L} + hB(t) = g_2(t)$$
$$\frac{-g_1(t) + B(t)}{L} + hB(t) = g_2(t)$$
$$-g_1(t) + B(t) + LhB(t) = Lg_2(t)$$
$$-g_1(t) + B(t)(1 + Lh) = Lg_2(t)$$
$$B(t)(1 + Lh) = Lg_2(t) + g_1(t)$$
$$B(t) = \frac{g_1(t) + Lg_2(t)}{1 + Lh}$$

Hence, we have

$$u(x,t) = g_1(t)\left[1 - \frac{x}{L}\right] + \frac{g_1(t) + Lg_2(t)}{1 + Lh}\left[\frac{x}{L}\right] + U(x,t)$$

and so if we substitute this into the original problem (7), we get our transformed problem in $U(x,t)$.

P.D.E.	$U_t = \alpha^2 U_{xx} - S_t$	(nonhomogeneous PDE)
B.C.s	$\begin{cases} U_x(L,t) + hU(L,t) = 0 \\ U(0,t) = 0 \end{cases}$	(homogeneous BCs)
I.C.	$u(x,0) = \phi(x) - S(x,0)$	(new IC-but known)

We now have our new problem with zero BCs (unfortunately, the PDE is nonhomogeneous). We can't solve this problem by separation of variables, but we will solve it by integral transforms and eigenfunction expansions.

NOTE:

For BCs of the form

$$u(0,t) = g_1(t)$$
$$u(L,t) = g_2(t)$$

the transformation is

$$u(x,t) = \left[g_1(t) + \frac{x}{L}\big(g_2(t) - g_1(t)\big)\right] + U(x,t)$$

Example: Transform

P.D.E.	$u_t = u_{xx}$	$0 < x < 1$	$0 < t < \infty$
B.C.s	$\begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases}$		$0 < t < \infty$
I.C.	$u(x,0) = x^2$		$0 \le x \le 1$

to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time?

Solution:

$$u(x,t) = \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

$k_1 = 0$, $k_2 = 1$ and $L = 1$

$$u(x,t) = x + U(x,t)$$

$$u_t = U_t,\ u_{xx} = U_{xx}$$

New

P.D.E.	$U_t = U_{xx}$	
B.C.s	$\begin{cases} 0 = u(0,t) = 0 + U(0,t) \Rightarrow U(0,t) = 0 \\ 1 = u(1,t) = 1 + U(1,t) \Rightarrow U(1,t) = 0 \end{cases}$	
I.C.	$x^2 = u(x,0) = x + U(x,0) \Rightarrow U(x,0) = x^2 - x$	

Thus we have

P.D.E.	$U_t = U_{xx}$	$0 < x < 1$	$0 < t < \infty$
B.C.s	$\begin{cases} U(0,t) = 0 \\ U(1,t) = 0 \end{cases}$		$0 < t < \infty$
I.C.	$U(x,0) = x^2 - x$		$0 \le x \le 1$

Sine the PDE and BCs are linear and homogeneous then this problem can be solve by separation of variables

$$U(x,t) = \sum_{i=1}^{\infty} \boldsymbol{A}_n e^{-(n\pi)^2 t} \, sin(n\pi x)$$

where

$$\boldsymbol{A}_n = 2 \int_0^1 (x^2 - x) \, sin(n\pi x) \, dx$$

The final solution is

$$u(x,t) = x + \sum_{i=1}^{\infty} \boldsymbol{A}_n e^{-(n\pi)^2 t} \, sin(n\pi x)$$

which is same solution for different values of time that we find in problem 5 in *LESSON* 5.

PROBLEMS

1. Solve the initial-boundary-value problem

   P.D.E. $u_t = \alpha^2 u_{xx}$ $0 < x < 1$ $0 < t < \infty$

   B.C.s $\begin{cases} u(0,t) = 1 \\ u_x(1,t) + hu(1,t) = 1 \end{cases}$ $0 < t < \infty$

   I.C. $u(x,0) = sin(\pi x) + x$ $0 \le x \le 1$

   by transforming it into homogeneous BCs and then solving the transformed problem.

2. Transform

   P.D.E. $u_t = u_{xx}$ $0 < x < 1$ $0 < t < \infty$

   B.C.s $\begin{cases} u_x(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 1 \end{cases}$ $0 < t < \infty$

   I.C. $u(x,0) = sin(\pi x)$ $0 \le x \le 1$

   into a new problem with zero BCs; is the new PDE homogeneous?

LESSON 5: Solving More Complicated Problems by Separation of Variables

Show how more complicated heat-flow problems can be solved by separation of variables.

We start with a one-dimensional heat-flow problem where one of the BCs contains derivatives.

Heat-Flow Problem with Derivative BC

Consider an apparatus

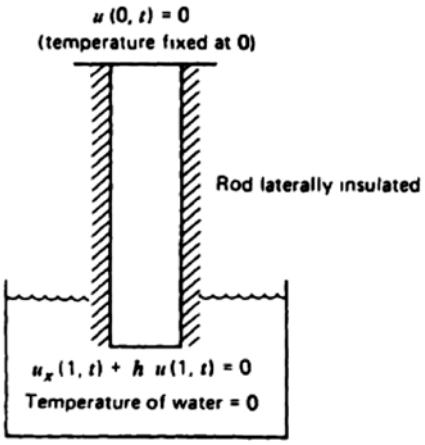


Diagram for the Initial-Boundary-Value Problem.

in which we fix the temperature at the top of the rod at $u(0,t)=0$ and immerse the bottom of the rod in a solution of water fixed at the same temperature of zero (zero refers to some reference temperature). The natural flow of heat (Newton's law of cooling) says that the BC at $x=1$ is

Suppose now the initial temperature of the rod is $u(x,0)=x$, but instantaneously thereafter $(t>0)$, we apply our BCs. To find the ensuing temperature, we must solve the IBVP

$$\begin{array}{lll} \text{P.D.E.} & u_t=\alpha^2 u_{xx} \qquad 0<x<1 & 0<t<\infty \\ \text{B.C.s} & \begin{cases} u(0,t)=1 \\ u_x(1,t)+hu(1,t)=0 \end{cases} & 0<t<\infty \\ \text{I.C.} & u(x,0)=x & 0\le x\le 1 \end{array} \qquad (8)$$

To apply the separation of variables method, we carry out the following steps:

STEP 1 (Separating the PDE into two ODEs)

Substituting $u(x,t)=X(x)T(t)$ into the PDE gives

$$XT'=\alpha^2 X''T$$

and dividing by $\alpha^2 XT$, we get

$$\frac{T'}{\alpha^2 T}=\frac{X''}{X}$$

Since the left-hand side depends *only* on time and the right-hand side depends only on x (and since x and t are independent), both sides of this equation must be constants. Setting them both equal to μ gives the two ODEs

$$T' - \mu\alpha^2 T = 0$$
$$X'' - \mu X = 0$$

We have now completed the *separation process.*

STEP 2 (Finding the separation constant)

First of all, μ must not be positive or else $T(t)$ will grow exponentially to infinity (which would make $u = XT$ go to infinity-which we can reject on physical grounds).

Secondly, suppose $\mu = 0$. This being the case, we have

$$X'' = 0$$

and thus

$$X(x) = A + Bx$$

But since the BCs of the problem are

$$\begin{matrix} u(0,t) = X(0)T(t) = 0 \\ u_x(1,t) + hu(1,t) = 0 \end{matrix} \Longrightarrow \begin{matrix} A = 0 \\ B = 0 \end{matrix}$$

which would mean $u(x,t) = 0$. In other words, $\mu = 0$ gives only $u = 0$; hence we throw it out (we are looking for nonzero solutions).

Finally, if $\mu < 0$, we call $\mu = -\lambda^2$ and write the two ODEs as

$$T' + \lambda^2\alpha^2 T = 0$$
$$\mathrm{X}'' + \lambda^2 \mathrm{X} = 0$$

which gives us solutions

$$T(t) = Ae^{-\lambda^2\alpha^2 t}$$
$$X(x) = B\,sin(\lambda x) + C\,cos(\lambda x)$$

Hence, what we have is that *any function*

$$u(x,t) = e^{-(\lambda\alpha)^2 t}[\boldsymbol{A}\,sin(\lambda x) + \boldsymbol{B}\,cos(\lambda x)]$$

for *any* λ and *any* $\boldsymbol{A}$ and $\boldsymbol{B}$ will satisfy the PDE (the reader can verify this calculation on his or her own). What we'd like to do now is find out how many of these functions will satisfy the BCs

$$u(0,t) = 1$$
$$u_x(1,t) + hu(1,t) = 0$$

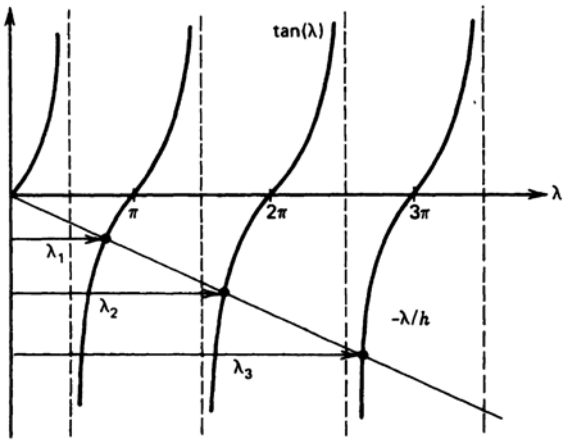
Substituting the BCs into the solution gives us conditions on λ, A, and B that must be satisfied; namely,

$$\boldsymbol{B}e^{-(\lambda\alpha)^2 t} = 0 \Rightarrow \boldsymbol{B} = 0$$
$$\boldsymbol{A}\lambda e^{-(\lambda\alpha)^2 t}\,cos\,\lambda + h\boldsymbol{A}e^{-(\lambda\alpha)^2 t}\,sin\,\lambda = 0$$

Performing a little algebra on this last equation gives us our desired condition on λ

$$\tan\lambda = -\frac{\lambda}{h}$$

In other words, to find λ, we must find the intersections of the curves $tan\ \lambda$ and $-\lambda/h$.



Graph showing intersections of $tan\ \lambda$ and $-\lambda/h$.

These values λ_1, λ_2,… can be computed numerically for a given λ on a computer and are called the ***eigenvalues*** of the boundary-value problem

$$X'' + \lambda^2 X = 0$$
$$X(0) = 0 \qquad (9)$$
$$X'(1) + hX(1) = 0$$

In other words, they are the values of λ for which there exists a *nonzero solution*. The eigenvalues λ_n which, in this case, are the roots of $\tan\lambda = -\lambda/h$, have been computed (for $h = 1$) numerically, and the first five values are listed in Table

n	λ_n
1	2.02
2	4.91
3	7.98
4	11.08
5	14.20

The solutions of (8) corresponding to the eigenvalues λ_n,. are called the **eigenfunctions** $X_n(x)$, and for this problem, we have

$$X_n(x) = sin(\lambda_n x)$$

STEP 3 (Finding the fundamental solutions)

We now have an infinite number of functions (fundamental solutions),

$$u(x,t) = X_n(x)T_n(t) = e^{-(\lambda\alpha)^2 t} sin(\lambda_n x)$$

each one satisfying the PDE and the BCs. The final step is to add these functions together (the sum will still satisfy the PDE and BCs, since both the PDE and BCs are linear and homogeneous) in such a way that they agree with the IC when $t = 0$; that is, we sum

$$u(x,t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} sin(\lambda_n x)$$

so that the IC $u(x,0) = x$ is satisfied. In other words,

$$u(x,t) = x = \sum_{n=1}^{\infty} a_n sin(\lambda_n x) \tag{9}$$

STEP 4 (Expansion of the IC as a sum of eigenfunctions)

To find the constants a_n in the eigenfunction expansion (9), we must multiply each side of the equation by $sin(\lambda_m x)$ and integrate *x* from 0 to 1; that is;

$$\begin{aligned}\int_0^1 \xi \, sin(\lambda_m \xi) d\xi &= \sum_{n=1}^{\infty} a_n \int_0^1 sin(\lambda_n \xi) \, sin(\lambda_m \xi) d\xi \\ &= a_n \int_0^1 sin^2(\lambda_n \xi) \, d\xi \\ &= a_m \left(\frac{\lambda_m - sin\, \lambda_m \, cos\, \lambda_m}{2\lambda_m} \right)\end{aligned}$$

Solving for a_m (we'll change the notation to a_n), we get our desired result

$$a_n = \left(\frac{2\lambda_n}{\lambda_n - sin\, \lambda_n \, cos\, \lambda_n} \right) \int_0^1 \xi \, sin(\lambda_n \xi) d\xi$$

In other words, our solution to (8) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

where the constants a_n are given above. In this problem, the first five constants a_n have been computed and are listed in Table

n	a_n
1	0.24
2	0.22
3	-0.03
4	-0.11
5	-0.09

Hence, the first three terms of solution for the IBVP (8) are

$$u(x,t) = 0.24e^{-4t} \sin(2x) + 0.22e^{-24t} \sin(4.9x) + 0.3e^{-63.3} \sin(7.98x) + \cdots$$

PROBLEMS

1. Solve the following heat-flow problem:

P.D.E.	$u_t = u_{xx}$	$0 < x < 1$	$0 < t < \infty$
B.C.s	$\begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases}$		$0 < t < \infty$
I.C.	$u(x,0) = x$		$0 \le x \le 1$

   by separation of variables. Does your solution agree with your intuition? What is the steady-state solution?

2. Solve the following problem with insulated boundaries:

P.D.E.	$u_t = u_{xx}$	$0 < x < 1$	$0 < t < \infty$
B.C.s	$\begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases}$		$0 < t < \infty$
I.C.	$u(x,0) = x$		$0 \le x \le 1$

   by separation of variables. Does your solution agree with your intuition? What is the steady-state solution?

3. Solve the following problem with insulated boundaries:

P.D.E.	$u_t = u_{xx}$	$0 < x < 1$	$0 < t < \infty$
B.C.s	$\begin{cases} u_x(0,t) = 0 \\ u(1,t) = 0 \end{cases}$		$0 < t < \infty$
I.C.	$u(x,0) = x$		$0 \le x \le 1$

   by separation of variables. Does your solution agree with your intuition? What is the steady-state solution?

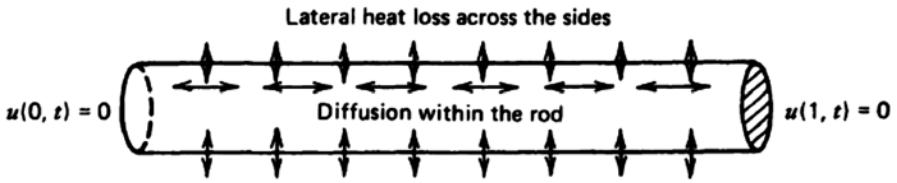
LESSON 6: Transforming Hard Equations into Easier Ones

Transforming a Heat-Flow Problem with Lateral Heat Loss into an Insulated Problem

Consider the following problem:

$$\begin{aligned} &\text{P.D.E.} && u_t = \alpha^2 u_{xx} - \beta u \quad 0 < x < 1 && 0 < t < \infty \\ &\text{B.C.s} && \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} && 0 < t < \infty \\ &\text{I.C.} && u(x,0) = \phi(x) && 0 \le x \le 1 \end{aligned} \tag{10}$$

where the term $-\beta u$ represents heat flow across the lateral boundary



Heat flow described by $u_t = \alpha^2 u_{xx} - \beta u$

The goal is to introduce a *new temperature* $w(x,t)$ in place of $u(x,t)$, so that the PDE in w is simpler than the original one

$$u_t = \alpha^2 u_{xx} - \beta u$$

This is a common technique in PDEs, and the transformation is generally based on an intuitive feeling of how the solution of the original PDE behaves. For example, in our problem (10), the temperature $u(x,t)$ at any point x_0 is changing as a result of two phenomena

1. *diffusion* of heat within the rod (due to $\alpha^2 u_{xx}$).
2. heat flow across the lateral boundary (due to $-\beta u$).

The important point is that if there were no diffusion within the rod ($\alpha = 0$), then the temperature at each point x_0 would "damp" exponentially to zero according to

$$u(x_0,t) = u(x_0,0)e^{-\beta t}$$

By means of this observation, we wonder if we can essentially decompose the temperature $u(x,t)$ of problem (10) into two factors

$$u(x,t) = e^{-\beta t} w(x,t) \tag{11}$$

or

$$\text{Noninsulated temperature} = e^{-\beta t}\ \text{(insulated temperature)}$$

where $w(x,t)$ would represent the temperature due to diffusion only. Let's see what happens if we substitute this expression into problem (10); this is a routine calculation (the reader can do it on his or her own), and we arrive at

$$\begin{array}{lll} \text{P.D.E.} & w_t = \alpha^2 w_{xx} \quad 0 < x < 1 & 0 < t < \infty \\ \text{B.C.s} & \begin{cases} w(0,t) = 0 \\ w(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{I.C.} & w(x,0) = \phi(x) & 0 \le x \le 1 \end{array} \qquad (12)$$

This is exactly the same problem we started with except that now the PDE doesn't contain $-\beta u$; so all we have to do to solve (10) is solve the transformed problem (12) and then multiply the solution $w(x,t)$ by $e^{-\beta t}$. In this case, we have already solved (12) previously by the separation of variables method and found

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

$$a_n = 2\int_0^1 \phi(\xi)\, sin(n\pi\xi)\, d\xi$$

and, hence, the solution of the original problem (10) is

$$u(x,t) = e^{-\beta t} w(x,t)$$

The following example is solved by this technique.

Example: solve the problem

$$\begin{array}{lll} \text{P.D.E.} & u_t = \alpha^2 u_{xx} - u \quad 0 < x < 1 & 0 < t < \infty \\ \text{B.C.s} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{I.C.} & u(x,0) = sin(\pi x) + 0.5 sin(3\pi x) & 0 \le x \le 1 \end{array}$$

by the preceding strategy, we

(a) Neglect the convection term $-u$ for the time being.

(b) Solve the initial-boundary-value problem without the term $-u$ to get:

$$u(x,t) = e^{-\pi^2 t} sin(\pi x) + 0.5 e^{-(3\pi)^2 t} sin(3\pi x)$$

(c) Multiply this solution by the convection factor $e^{-\beta t} = e^{-t}$ to get the solution

$$u(x,t) = e^{-t}\left[e^{-\pi^2 t} sin(\pi x) + 0.5 e^{-(3\pi)^2 t} sin(3\pi x)\right]$$

The diffusion-convection equation

$$u_t = \alpha^2 u_{xx} - \nu u_x$$

(ν is a constant) can also be transformed to

$$w_t = \alpha^2 w_{xx}$$

In this case, the transformation is

$$u(x,t) = e^{\nu(x - \nu t/2)/2\alpha^2} w(x,t)$$

This transformation essentially factors out the part of the solution (exponential factor) that is due to the moving medium.

PROBLEMS

1. Solve the the diffusion problem

   P.D.E. $u_t = \alpha^2 u_{xx} - u_x \quad 0 < x < 1 \quad 0 < t < \infty$

   B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$

   I.C. $u(x,0) = e^{x/2} \quad 0 \le x \le 1$

   by transforming it into an easier problem.

2. Solve the problem

   P.D.E. $u_t = \alpha^2 u_{xx} - u + x \quad 0 < x < 1 \quad 0 < t < \infty$

   B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} \quad 0 < t < \infty$

   I.C. $u(x,0) = 0 \quad 0 \le x \le 1$

   by

   (a) changing the nonhomogeneous BCs to homogeneous ones.

   (b) transforming into a new equation without the term -u.

   (c) solving the resulting problem.

3. Solve

   P.D.E. $u_t = \alpha^2 u_{xx} - u \quad 0 < x < 1 \quad 0 < t < \infty$

   B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$

   I.C. $u(x,0) = \sin(\pi x) \quad 0 \le x \le 1$

   directly by separation of variables without making any preliminary transformation. Does your solution agree with the solution you would obtain if the transformation

   $$u(x,t) = e^{-t} w(x,t)$$

   were made in advance?

***LESSON* 7: Solving Nonhomogeneous PDEs (Eigenfunction Expansions)**

Consider the nonhomogeneous problem

$$\begin{array}{lll} \text{P.D.E.} & u_t = \alpha^2 u_{xx} + f\,(x,t) \quad 0 < x < 1 & 0 < t < \infty \\ \text{B.C.s} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{I.C.} & u(x,0) = \phi(x) & 0 \le x \le 1 \end{array} \tag{13}$$

To solve this problem, we divide the procedure into the following steps:

STEP 1 The basic idea in this method is to decompose the heat source $f(x,t)$ into simple components

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \cdots + f_n(t)X_n(x) + \cdots$$

and find the response $u_n(x,t)$ to each of these individual components $f_n(t)X_n(x)$. The solution to our problem is then

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

To determine how to decompose $u(x,t)$ into its component parts $T_n(t)X_n(x)$ is one of the major problems. It turns out that the $X_n(x)$ factors in this problem are the eigenvectors of the Sturm-Liouville system we get when solving the associated homogeneous problem to (13) by separation of variables; that is,

$u_t = \alpha^2 u_{xx}$ (note that $f(x,t) = 0$)
$u(0,t) = 0$
$u(1,t) = 0$
$u(x,0) = \phi(x)$

in this case, the Sturm-Liouville problem we find when separating variables is

$$X'' + \lambda^2 X = 0$$
$$X(0) = 0$$
$$X(1) = 0$$

and, hence, the $X_n(x)$ are

$$X_n(x) = sin(n\pi x) \qquad n = 1,2, \ldots$$

Hence, our decomposition of the heat source has the form

$$f(x,t) = f_1(t)\, sin(\pi x) + f_2(t)\, sin(2\pi x) + \cdots + f_n(t)\, sin(n\pi x) + \cdots$$

Finally, to find the functions $f_n(t)$, we merely multiply each side of this equation by $sin(m\pi x)$ and integrate from zero to one (with respect to *x*); hence, we have

$$\int_0^1 f(x,t)\, sin(m\pi x)\; dx = \sum_{n=0}^{\infty} f_n(t) \int_0^1 sin(m\pi x)\, sin(n\pi x)\; dx$$
$$= \frac{1}{2} f_m(t)$$

or (changing *m* to *n*)

$$f_n(t) = 2\int_0^1 f(x,t)\,sin(n\pi x)\;dx$$

This will give us an equation for the coefficients $f_n(t)$ in terms of the heat source $f(x,t)$.

STEP 2 (Find the response $u_n(x,t) = T_n(t)X_n(x)$ to input $f_n(\mathrm{t})X_n(x)$)

We now replace the heat source $f(x,t)$ by its decomposition

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\,sin(n\pi x)$$

and then find the individual responses

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\,sin(n\pi x)$$

Substituting $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\,sin(n\pi x)$ into the system

$$u_t = \alpha^2 u_{xx} + \sum_{n=1}^{\infty} f_n(t)\,sin(n\pi x)$$
$$u(0,t) = 0$$
$$u(1,t) = 0$$
$$u(x,0) = \phi(x)$$

gives us

$$\sum_{n=1}^{\infty} T'_n(t)\,sin(n\pi x) = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n(t)\,sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\,sin(n\pi x)$$

$$u(0,t) = \sum_{n=1}^{\infty} T_n(t)\,sin(0) = 0 \quad nothing\ 0 = 0$$

$$u(1,t) = \sum_{n=1}^{\infty} T_n(t)\,sin(n\pi) = 0 \;\; nothing\ 0 = 0$$

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\,sin(n\pi x) = \phi(x)$$

Rewriting the PDE and the IC as

$$\text{PDE} \quad \sum_{n=1}^{\infty} \left(T'_n(t) + (\alpha n\pi)^2 T_n(t) - f_n(t)\right) sin(n\pi x) = 0$$

$$\text{IC} \quad \sum_{n=1}^{\infty} T_n(0)\,sin(n\pi x) = \phi(x)$$

we can see fairly easily that $T_n(t)$ will satisfy the simple initial value problem

$$T'_n(t) + (\alpha n\pi)^2 T_n(t) = f_n(t)$$
$$T_n(0) = 2\int_0^1 \phi(x)\, sin(n\pi x)\, dx = a_n$$

This ODE is one of the easier ones to solve (use an integrating factor) and has the solution

$$T_n(t) = a_n e^{-(\alpha n\pi)^2 t} + \int_0^t e^{-(\alpha n\pi)^2 (t-\tau)} f_n(\tau) d\tau$$

Hence, the solution of problem is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\, sin(n\pi x)$$
$$= \sum_{n=1}^{\infty} a_n e^{-(\alpha n\pi)^2 t}\, sin(n\pi x) + \sum_{n=1}^{\infty} \left[sin(n\pi x) \int_0^t e^{-(\alpha n\pi)^2 (t-\tau)} f_n(\tau) d\tau \right]$$

This completes the problem. We will show how this method can be applied to a specific example.

Solution of a Problem by the Eigenfunction-Expansion Method

Consider the simple problem

P.D.E. $u_t = \alpha^2 u_{xx} + sin(3\pi x) \quad 0 < x < 1 \quad 0 < t < \infty$

B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$

I.C. $u(x,0) = sin(\pi x) \quad 0 \le x \le 1$

Our goal is to compute the coefficients $T_n(t)$ in the expansion

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\, sin(n\pi x)$$

(the eigenfunctions $X_n(x)$ are still the same for this problem).

If we substitute this expansion in the problem, we will get an ODE for the functions $T_n(t)$ In fact, we will get

$$T'_n + (n\pi\alpha)^2 T_n = f_n(t) = 2\int_0^1 sin(3\pi x)\, sin(n\pi x)\, dx = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases}$$

$$T_n(0) = 2\int_0^1 sin(\pi x)\, sin(n\pi x)\, dx = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Writing out these equations for $n = 1,2,$ ... we see

$$(n=1)\ \left.\begin{matrix} T'_1 + (\pi\alpha)^2 T_1 = 0 \\ T_1(0) = 1 \end{matrix}\right\} \Longrightarrow T_1(t) = e^{-(\pi\alpha)^2 t}$$

$$(n=2)\ \left.\begin{matrix} T'_2 + (\pi\alpha)^2 T_2 = 0 \\ T_2(0) = 0 \end{matrix}\right\} \Longrightarrow T_2(t) = 0$$

$$(n=3)\ \left.\begin{matrix} T'_3 + (\pi\alpha)^2 T_3 = 1 \\ T_3(0) = 0 \end{matrix}\right\} \Longrightarrow T_3(t) = \frac{1}{(3\pi\alpha)^2}\left[1 - e^{-(3\pi\alpha)^2 t}\right]$$

$$(n \geq 4)\ \left.\begin{matrix} T'_n + (\pi\alpha)^2 T_n = 0 \\ T_n(0) = 0 \end{matrix}\right\} \Longrightarrow T_n(t) = 0$$

Hence the solution of our problem is

$$u(x,t) = e^{-(\pi\alpha)^2 t}\, sin(\pi x) + \frac{1}{(3\pi\alpha)^2}\left[1 - e^{-(3\pi\alpha)^2 t}\right] sin(3\pi x)$$

PROBLEMS

1- Solve the problem

P.D.E. $u_t = u_{xx} + sin(\pi x) + sin(2\pi x) \quad 0 < x < 1 \quad 0 < t < \infty$

B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$

I.C. $u(x,0) = 0 \quad 0 \leq x \leq 1$

by the method of eigenfunction expansion.

2- Solve the problem

P.D.E. $u_t = u_{xx} + sin(\pi x) \quad 0 < x < 1 \quad 0 < t < \infty$

B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$

I.C. $u(x,0) = 1 \quad 0 \leq x \leq 1$

by the method of eigenfunction expansion.

3- Solve the problem

P.D.E. $u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$

B.C.s $\begin{cases} u(0,t) = 0 \\ u(1,t) = cos\, t \end{cases} \quad 0 < t < \infty$

I.C. $u(x,0) = x \quad 0 \leq x \leq 1$

by

(a) Transforming it to one with zero BCs.

(b) Solving the resulting problem by expanding it in terms of eigenfunctions.

***LESSON* 8: Integral Transforms (Sine and Cosine Transforms)**

The idea of integral transforms is transform PDEs in n variables into differential equations in $n-1$ variables.

An integral transformation is merely a transformation that assigns to one function $f(t)$ a new function $F(s)$ by means of a formula like

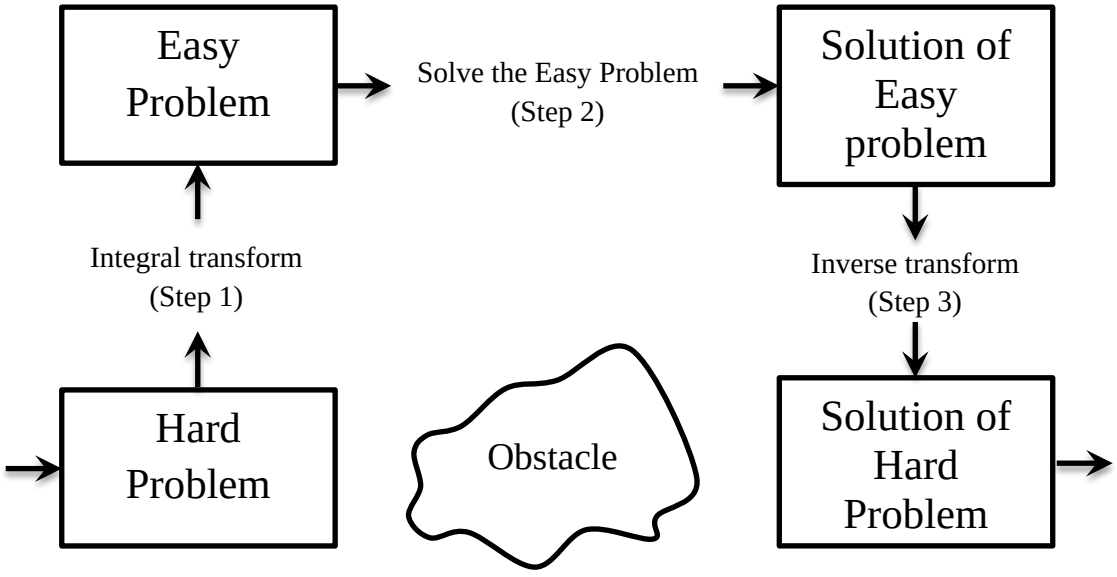
$$F(s) = \int_A^B K(s,t)\, f(t) dt$$

Note that we start with a function of t and end with a function of s. The function $K(s,t)$ is called the kernel of the transformation and is the major ingredient that distinguishes one transform from another; it is chosen so that the transform has certain desirable properties. The limits of integration A and B also change from transformation to transformation.

The general philosophy behind integral transformations is that they eliminate *partial derivatives* with respect to one of the variables; hence, the new equation has one less variable. For example, if we apply a transform to the PDE

$$u_t = u_{xx}$$

for the purpose of eliminating the time derivative, then we would arrive at an ODE in *x*. Integral transforms change problems into easier ones. The transformed problem is then solved, and its inverse is obtained to find the solution to the original problem;



we see that along with every integral transform, there is an *inverse transform* that will reproduce that original function from its transform. The transform and its inverse together form what is called a transform pair.

Some Common Transform Pairs

Transform pairs

1. $\begin{cases}\mathcal{F}_s[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)sin(\omega t)\,dt & \text{(Fourier sine transform)}\\ \mathcal{F}_s^{-1}[F] = f(t) = \int_0^\infty F(\omega)sin(\omega t)\,d\omega & \text{(inverse sine transform)}\end{cases}$
2. $\begin{cases}\mathcal{F}_c[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)cos(\omega t)\,dt & \text{(Fourier cosine transform)}\\ \mathcal{F}_c^{-1}[F] = f(t) = \int_0^\infty F(\omega)cos(\omega t)\,d\omega & \text{(inverse cosine transform)}\end{cases}$
3. $\begin{cases}\mathcal{F}[f] = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-i\omega x}\,dx & \text{(Fourier transform)}\\ \mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\omega)e^{i\omega x}\,d\omega & \text{(inverse Fourier transform)}\end{cases}$
4. $\begin{cases}\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}\,dt & \text{(Laplace transform)}\\ \mathcal{L}^{-1}[F] = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds & \text{(inverse Laplace transform)}\end{cases}$

Sine and Cosine Transforms of Derivatives

1. $\mathcal{F}_s[f'] = \omega\mathcal{F}_c[f]$ (proved by integration by parts)
2. $\mathcal{F}_s[f''] = \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s[f]$
3. $\mathcal{F}_c[f'] = -\frac{2}{\pi}f(0) + \omega\mathcal{F}_s[f]$
4. $\mathcal{F}_c[f''] = -\frac{2}{\pi}\omega f'(0) - \omega^2\mathcal{F}_c[f]$

Solution of an Infinite-Diffusion Problem via the Sine Transform

The problem we are interested in is the *infinite diffusion problem*

P.D.E. $u_t = \alpha^2 u_{xx}$ $\quad 0 < x < \infty \quad 0 < t < \infty$

B.C.s $u(0,t) = A$ $\quad 0 < t < \infty$

I.C. $u(x,0) = 0$ $\quad 0 \le x < \infty$

STEP 1 To solve this, we break it into three simple steps. First our strategy is to transform the *x-variable* via the Fourier sine transform so that we get an ODE in time. We start by transforming each side of the PDE; in other words

$$\mathcal{F}_s[u_t] = \alpha^2\mathcal{F}_s[u_{xx}]$$

Let's consider each term individually:

$\mathcal{F}_s[u_t]$: The partial derivative *u*, in this problem is what we could call the *off derivative*, since our transform is *with respect to x.*

$$\begin{aligned}\mathcal{F}_s[u_t] &= \frac{2}{\pi}\int_0^\infty u_t(x,t)\,sin(\omega x)dx\\ &= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int_0^\infty u(x,t)\,sin(\omega x)dx\right]\\ &= \frac{d}{dt}\mathcal{F}_s[u]\\ &= \frac{d}{dt}U(t)\end{aligned}$$

u is a function of *x* and *t*, whereas its transform

$$\mathcal{F}_s[u] = U(\omega, t)$$

is a function of ω and t. The new variable ω will be treated like a parameter in the new problem, and, hence, we call the sine transform a function of t alone.

$$\mathcal{F}_s[u] = U(t)$$

We have the identity

$$\mathcal{F}_s[u_{xx}] = \frac{2}{\pi}\omega u(0,t) - \omega^2 \mathcal{F}_s[u]$$
$$= \frac{2}{\pi}\omega A - \omega^2 U(t)$$

Substituting these expressions into our PDE, we arrive at the ODE

$$\frac{dU(t)}{dt} = \alpha^2\left[-\omega^2 U(t) + \frac{2\omega A}{\pi}\right]$$

The only thing missing is an IC for $U(t)$; we get this by transforming the IC $u(0,t) = 0$ to get

$$\mathcal{F}_s[u(0,t)] = U(0) = 0$$

This completes the first step in the transform process-we have changed the original problem into an initial-value problem

O.D.E. $\frac{dU}{dt} + \omega^2\alpha^2 U = \frac{2A\omega\alpha^2}{\pi}$

I.C. $U(0) = 0$

STEP 2 To solve this IVP, we could use a variety of elementary techniques from ordinary differential equations (integrating factor, homogeneous and particular solution); in any case, the solution is

$$U(t) = \frac{2A}{\pi\omega}\left(1 - e^{-\omega^2\alpha^2 t}\right)$$

We have now found the sine transform for the answer $u(x,t)$. The last step is to find the inverse transform of $U(t)$; that is,

$$u(x,t) = \mathcal{F}_s^{-1}[U]$$

STEP 3 To find the solution, we can either evaluate the inverse transform directly from the integral or else resort to the tables.

Using the tables, we see that

$$u(x,t) = A\ erfc\left(x/2\alpha\sqrt{t}\right)$$

where $erfc(x), 0 < x < \infty$ is called the **complementary-error function** and is given by

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2} dt$$

PROBLEMS

1- Solve by means of the sine or cosine transform

P.D.E. $u_t = \alpha^2 u_{xx}$ $\quad 0 < x < \infty \quad 0 < t < \infty$

B.C.s $u(0,t) = 0$ $\quad 0 < t < \infty$

I.C. $u(x,0) = H(1-x)$ $\quad 0 \le x < \infty$

where $H(1-x)$ is the *Heaviside function:*

$$H(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

2- Verify the *orthogonality* properties of sines and cosines on the interval $[-L, L]$

$$\int_{-L}^{L} \sin(m\pi x/L)\sin(n\pi x/L)\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$$\int_{-L}^{L} \sin(m\pi x/L)\cos(n\pi x/L)\,dx = 0 \;\; all\; m, n = 1,2,3,\ldots$$

***LESSON* 9: The Fourier Transform and Its Application to PDEs**

We shall illustrate several useful properties of the Fourier transform and to show how these properties can be used to solve PDEs.

In particular, it is shown how the Fourier transform changes *differentiation* to *multiplication*, so differential equations change into algebraic equations.

Also, the idea of the *infinite convolution* is introduced.

The Fourier transform of the function $f(x)$ for $-\infty < x < \infty$ is given by the formula

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \qquad (14)$$

That is, we start with a function $f(x)$ defined on the real x-axis, substitute it into equation (14), and arrive at the new function $F(\xi)$ for $-\infty < \xi < \infty$. For example, we list some common Fourier transforms.

	Function $f(x)$	Fourier transforms $F(\xi)$
1	$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ -e^{-x}, & x < 0 \end{cases}$	$F(\xi) = -i\sqrt{\frac{2}{\pi}} \frac{\xi}{1+\xi^2}$ $complex\ funcion$
2	$f(x) = \begin{cases} 1, & -1 < x < 0 \\ 0, & elsewhere \end{cases}$	$F(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi}$ $real\ funcion$
3	$f(x) = e^{-x^2}$	$F(\xi) = \frac{1}{\sqrt{2}} e^{-\left(\frac{\xi}{2}\right)^2}$ $real\ funcion$

Useful Properties of the Fourier Transform

***Property* 1** (Fourier Transform Pair)

The Fourier transform of $f(x)$, $-\infty < x < \infty$, produces a new function $F(\xi)$ defined by the formula

$$\mathcal{F}[f] = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

and the inverse Fourier transform of $F(\xi)$, $-\infty < \xi < \infty$, will reproduce the original function $f(x)$ according to`

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega$$

For example,

$$e^{-|x|} \xrightarrow{\mathcal{F}} \sqrt{\frac{2}{\pi}} \frac{\xi}{1+\xi^2} \xrightarrow{\mathcal{F}^{-1}} e^{-|x|}$$

Property 2 (Linear Transformation)

The Fourier transform is a linear transformation; that is

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

For example, the Fourier transform of the expression

$$\frac{1}{x^2+1} + 3e^{-x^2}$$

would be

$$\mathcal{F}\left[\frac{1}{x^2+1} + 3e^{-x^2}\right] = \mathcal{F}\left[\frac{1}{x^2+1}\right] + 3\mathcal{F}\left[e^{-x^2}\right]$$

Property 3 (Transformation of Partial Derivatives)

The Fourier transform transforms the x-variable (the variable of integration in the transform) and *if* the function being transformed is a partial derivative of a function $u(x,t)$ *with respect to* x, then the rules of transformation are

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(x,t) e^{-i\xi x}\, d\xi = i\xi \mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx}(x,t) e^{-i\xi x}\, d\xi = -\xi^2 \mathcal{F}[u]$$

On the other hand, if we transform the partial derivative $u_t(x,t)$ (and if the variable of integration in the transform is x), then the transform is given by

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t) e^{-i\xi x}\, dx = \frac{\partial}{\partial t} \mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{tt}(x,t) e^{-i\xi x}\, dx = \frac{\partial^2}{\partial t^2} \mathcal{F}[u]$$

Property 4 (Convolution Property)

Every integral transform has what is called a *convolution* property. The general idea is that the transform of a product of two functions $f(x)g(x)$ is *not* the product of the individual transforms; that is,

$$\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f(x)]\mathcal{F}[g(x)]$$

However, in transform theory there is something called the convolution $f * g$ of two functions that more or less plays the role of the product. What is true about this convolution $f * g$ is that

$$\mathcal{F}[f * g\,] = \mathcal{F}[f]\mathcal{F}[g]$$

So the convolution of $f * g$ is given by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi$$

From the definition of the convolution that given two functions $f(x)$ and $g(x)$, the convolution $(f * g)(x)$ is a new function.

Example of a Convolution of Two Functions

Given the two functions

$$f(x) = x$$
$$g(x) = e^{-x^2}$$

the convolution is given by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\xi) e^{-\xi^2}\, d\xi$$
$$= \frac{x}{\sqrt{2}} \qquad (a\ new\ function)$$

We have used the formula

$$\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\pi}$$

The importance of the convolution in applications is due to the fact that quite often, the final step in solving a PDE boils down to finding the inverse transform of some expression that we can interpret as the product of two transforms $\mathcal{F}[f]\mathcal{F}[g]$; that is, we must find

$$\mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\}$$

By taking the inverse of each side of

$$\mathcal{F}[f * g\,] = \mathcal{F}[f]\mathcal{F}[g]$$

We have

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\}$$

we have to do is find the inverse transform of *each* factor to get *f* and *g* and *then* compute their convolution.

Solution of an lnltlal-Value Problem

Consider

P.D.E. $u_t = \alpha^2 u_{xx}$ $\quad -\infty < x < \infty \quad 0 < t < \infty$

I.C. $u(x,0) = \phi(x)$ $\quad -\infty < x < \infty$

There are three basic steps in solving this problem

STEP 1 (Transforming the problem)

Since the space variable *x* ranges from $-\infty$ to ∞, we take tile Fourier transform of the PDE and IC with respect to this variable *x* (the variable of integration in the transform is *x*). Doing this, we get

$$\mathcal{F}[u_t\ ] = \alpha^2\mathcal{F}[u_{xx}]$$
$$\mathcal{F}[u(x,0)\ ] = \mathcal{F}[\phi(x)]$$

and using the properties of the Fourier transform, we have

$$\frac{dU(t)}{dt} = -\alpha^2\xi^2 U(t)$$

$U(0) = \Phi(\xi)$ (Φ is the Fourier transform of ϕ)

where $U(t) = \mathcal{F}[u(x,t)]$. The function U actually depends on *both* t and the new transformed variable ξ. but, for simplicity, since ξ is a constant insofar as the differential equation is concerned, we will drop the notation and just call $U = U(t)$.

STEP 2 (Solving the transformed problem)

Remember the new variable ξ is nothing more than a constant in this differential equation, so the solution to this problem is

$$U(t) = \Phi(\xi)e^{-\alpha^2\xi^2 t}$$

STEP 3 (Finding the inverse transform)

To find the solution $u(x,t)$, we merely compute

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U(\xi,t)] \\ &= \mathcal{F}^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}] \end{aligned}$$

Here is where the convolution $f * g = \mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\}$ comes to the rescue.

Using this property, we can write

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}] \\ &= \mathcal{F}^{-1}[\Phi(\xi)] * \mathcal{F}^{-1}\left[e^{-\alpha^2\xi^2 t}\right] \\ &= \phi(x) * \left[\tfrac{1}{\alpha\sqrt{2t}}e^{-(x^2/4\alpha^2 t)}\right] \qquad \text{(using tables)} \\ &= \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty}\phi(\xi)e^{-(x-\xi)^2/4\alpha^2 t}\,d\xi \end{aligned}$$

is the solution to our problem.

NOTES

The major drawback of the Fourier transform is that all functions cannot be transformed; for example, even simple functions like

$$f(x) = constant$$
$$f(x) = e^x$$
$$f(x) = \sin x$$

cannot be transformed, since the integral

$$\mathcal{F}[f] = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$$

does not exist. Only functions that damp to zero sufficiently fast as $|x| \longrightarrow \infty$ have transforms. Also, the Fourier transform could not be used to transform the time variable in the previous initial value problem, since $0 < t < \infty$.

PROILEMS

1- Solve the initial-value problem

P.D.E. $u_t = \alpha^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty$

I.C. $u(x,0) = e^{-x^2} \qquad -\infty < x < \infty$

by using the Fourier transform.

2- Verify the properties (HINT Use integration by parts)

$$\mathcal{F}[u_x] = i\xi\mathcal{F}[u]$$
$$\mathcal{F}[u_{xx}] = -\xi^2\mathcal{F}[u]$$

3- Verify that the convolution of two functions f and g can be written as either

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$$

or

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi$$

***LESSON* 10: The Laplace Transform**

The Laplace transform

$$\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

It is a very powerful tool for transforming initial-value problems in ODE into algebraic equations.

Not only is the Laplace transform useful in transforming ODEs into algebraic equations, but now we will use the Laplace transform to transform PDEs into ODEs.

The major difference in applying the Laplace transform to PDEs in contrast to ODEs is that now when the original PDE is transformed, the new resulting equation will be either a new PDE with one less independent variable or else an ODE in one variable.

Properties of the Laplace Transform

Property 1 (*Transform Pair*)

The Laplace transform and its inverse are given by

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{(Laplace transform)}$$

$$\mathcal{L}^{-1}[F] = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds \qquad \text{(inverse Laplace transform)}$$

The Laplace transform has one major advantage over the Fourier transform in that the damping factor e^{-st} in the integrand allows us to transform a wider class of functions (the factor $e^{-i\xi x}$ in the Fourier transform doesn't do any damping, since its absolute value is one).

Remark: (*Sufficient Conditions to Insure the Existence of a Laplace Transform*)
If

1. f is piecewise continuous on the interval $0 \le t \le A$ for any positive A.
2. we can find constants M and a such that $|f(t)| \le Me^{at}$ for all values of t greater than some number T

then the Laplace transform

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

exists for $s > a$.

We now list a few functions that have Laplace transforms and graph them on the s-axis.

	Function $f(x)$	Laplace transforms $F(s)$
1	$f(t) = 1, 0 < t < \infty$ Put $M = 1$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \frac{1}{s}$
2	$f(t) = e^{2t}$ Put $M = 1, a = 2$	$F(s) = \frac{1}{\mathrm{s} - 2}$
3	$f(t) = sin(wt)$ Put $M = 1, a = 0$	$F(s) = \frac{1}{s^2 + w^2}$
4	$f(t) = e^{t^2}$	Doesn't have a Laplace transform

***Property* 2 (*Transforms of Partial Derivatives*)**

Suppose we have a function $u(x,t)$ of two variables and wish to transform various partial derivatives u_t, u_{tt}, u_x, u_{xx},… Since the Laplace transform transforms the *t-variable* (variable of integration), the rules of transformation for partial derivatives are

$$\mathcal{L}[u_t] = \int_0^\infty u_t(x,t)e^{-st}\,dt = sU(x,s) - u(x,0)$$

$$\mathcal{L}[u_{tt}] = \int_0^\infty u_{tt}(x,t)e^{-st}\,dt = s^2U(x,s) - su(x,s) - u_t(x,0)$$

$$\mathcal{L}[u_x] = \int_0^\infty u_x(x,t)e^{-st}\,dt = \frac{\partial}{\partial x}U(x,s)$$

$$\mathcal{L}[u_{xx}] = \int_0^\infty u_{xx}(x,t)e^{-st}\,dt = \frac{\partial^2}{\partial x^2}U(x,s)$$

where $U(x,s) = \mathcal{L}[u(x,t)]$. The transform rules for u_x and u_{xx} are a result of a basic rule in calculus

$$\frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{df}{dx}(x,y)dy$$

while the rules for u_t and u_{tt} can be proven by using the integration by parts formula.

Property 3 (*Convolution Property*)

Convolution plays the same role here as it did in the Fourier transform. The finite convolution of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

(these two integrals are the same).

An example of the finite convolution of two functions

$$f(t) = t, g(t) = t$$

would be

$$(f * g)(t) = \int_0^t \tau(t-\tau)\, d\tau = \frac{t^3}{6}$$

As in the case of the infinite convolution, the important property of this new convolution is that

$$\mathcal{L}[f * g\,] = \mathcal{L}[f]\mathcal{L}[g]$$

or the equivalent formula

$$\mathcal{L}^{-1}\{\mathcal{L}[f]\mathcal{L}[g]\} = f * g$$

This property will allow us to find the inverse Laplace transform of a product of two functions (which we interpret as $\mathcal{L}[f]\mathcal{L}[g]$) by finding the inverses of each factor $\mathcal{L}[f]$ and $\mathcal{L}[g]$ to get f and g and then finding their convolution. For example

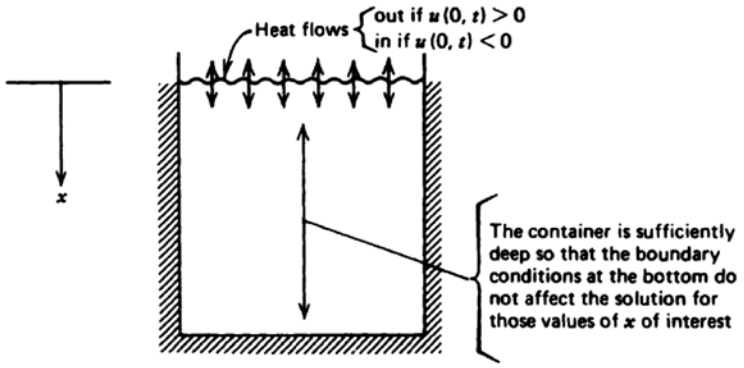
$$\mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+1}\right] = \int_0^t sin(\tau)\, d\tau = 1 - cos\, t$$

$$F(s) = \frac{1}{s} \xrightarrow{\mathcal{L}^{-1}} f(t) = 1 \qquad G(s) = \frac{1}{s^2+1} \xrightarrow{\mathcal{L}^{-1}} g(t) = sin\, t$$

Heat Conduction in a Semi Infinite Medium

Consider a large (deep) container of liquid that is insulated on the sides. Suppose the liquid has an initial temperature of u_0 and that the temperature of the air above the liquid is zero (some reference temperature). Our goal is to find the temperature of the liquid at various depths of the container at different values of time. To do so, we must solve the problem

P.D.E.	$u_t = u_{xx}$	$0 < x < \infty$	$0 < t < \infty$
B.C.s	$u_x(0,t) - u(0,t) = 0$		$0 < t < \infty$
I.C.	$u(x,0) = u_0$		$0 \le x < \infty$



To solve this problem, we transform the time variable t by means of the Laplace transform (conceivably, we could also transform x by means of the Laplace

transform, since x also ranges from 0 to ∞). Transforming our problem, we arrive at an ODE in x.

Using the second property of Laplace transform P.D.E., B.C. and I.C.

$$\mathcal{L}[u_t] = sU(x,s) - u(x,0) = sU(x) - u_0$$

$$\mathcal{L}[u_{xx}] = \frac{d^2}{dx^2}U(x)$$

$$\mathcal{L}[u_x] = \frac{d}{dx}U(x), at\ x = 0 \Longrightarrow \frac{d}{dx}U(x)\Big|_{x=0} = \frac{d}{dx}U(0)$$

$$\mathcal{L}[u(x,t)] = U(x), at\ x = 0 \Longrightarrow U(x)|_{x=0} = U(0)$$

Then the new ODE in x is

O.D.E. $sU(x) - u_0 = \frac{d^2}{dx^2}U(x)$

I.C. $\frac{d}{dx}U(0) = U(0)$

we first find the general solution (homogeneous + a particular solution), which is

$$U(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{u_0}{s}$$

Substituting this expression into the I.C. allows us to find the constants c_1 and c_2 (first note that $c_1 = 0$ or else the solution will go to infinity as x gets large).

To find c_2 from I.C.at $x = 0$ after driving $U(x)$ with respect to x.

$$-c_2\sqrt{s} = c_2 + \frac{u_0}{s}$$

$$c_2 + c_2\sqrt{s} = -\frac{u_0}{s} \Longrightarrow c_2\left(1+\sqrt{s}\right) = -\frac{u_0}{s} \Longrightarrow c_2 = -\frac{u_0}{s\left(1+\sqrt{s}\right)}$$

gives us the answer for $U(x)$

$$U(x) = -\frac{u_0 e^{-\sqrt{s}x}}{s\left(1+\sqrt{s}\right)} + \frac{u_0}{s}$$

Now for the last step. To find $u(x,t)$, we compute

$$u(x,t) = \mathcal{L}^{-1}[U(x)]$$

To find this inverse transform, we must resort to the tables of inverse Laplace transforms, they will give us

$$u(x,t) = u_0 - u_0\left[erfc\left(\frac{x}{2\sqrt{t}}\right) - erfc\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)e^{(x+t)}\right]$$

where

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}d\xi$$

is the *complementary-error function*.

PROBLEMS

1. Verify the following formula for the transform of the partial derivative u_t
$$\mathcal{L}[u_t] = sU(x,s) - u(x,0)$$
2. Solve the following initial-value problem by means of the Laplace transform
P.D.E. $u_t = \alpha^2 u_{xx}$ $\quad -\infty < x < \infty \quad 0 < t < \infty$
I.C. $u(x,0) = \sin x$ $\quad -\infty < x < \infty$
3. Solve the problem
P.D.E. $u_t = u_{xx}$ $\quad 0 < x < \infty \quad 0 < t < \infty$
B.C. $u(0,t) = \sin t$ $\quad 0 < t < \infty$
I.C. $u(x,0) = 0$ $\quad 0 \le x < \infty$
by means of the Laplace transform (transform *t*).

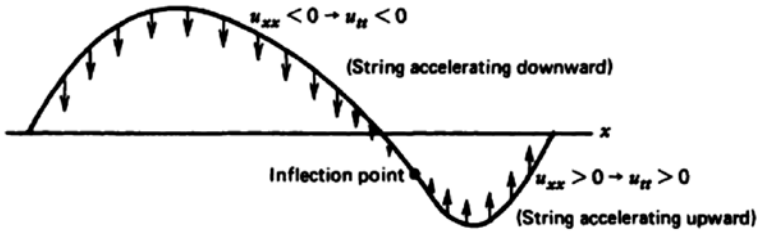
LESSON 11: The One-Dimensional Wave Equation (Hyperbolic Equations)

Interpretation of the Wave Equation

The one-dimensional wave equation is given by

$$u_{tt} = \alpha^2 u_{xx}$$

This equation is describing the vibration of a violin string. The expression u_{tt} represents the vertical acceleration of the string at a point *x*. this equation can be interpreted as saying that the acceleration of each point of the string is due to the tension in the string and that the larger the concavity u_{xx}.



The wave equation $u_{tt} = \alpha^2 u_{xx}$, also describes *longitudinal* and *torsional* vibrations of a rod. In these vibrations, the displacement is *parallel* to the rod (like hitting the end of the rod with a hammer), and $u(x,t)$ stands for the longitudinal displacement of a point on the rod from its equilibrium position.

Since the wave equation $u_{tt} = \alpha^2 u_{xx}$ contains a second-order time derivative u_{tt}, it requires two initial conditions

$u(x,0) = f(x)$ (initial position of the string)

$u_t(x,0) = g(x)$ (initial velocity of the string)

in order to uniquely define the solution for $t > 0$. This is in contrast to the heat equation, where only one IC was required.

The D'Alembert Solution of the Wave Equation

We start by solving the one-dimensional wave equation in free space. That is, the pure initial-value problem

$$\text{P.D.E.} \quad u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

$$\text{I.C.} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad -\infty \le x < \infty$$

This problem (which has no boundaries) describes the motion of an infinite string with given initial conditions and was solved in about 1750 by the French mathematician D' Alembert.

This so-called D' Alembert solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

is easy to compute once we have the two initial conditions. In addition, this solution has interesting interpretations in terms of moving wave motion.

D'Alembert's Solution to the One-Dimensional Wave Equation

We solve problem

$$\text{P.D.E.} \quad u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty \qquad (15)$$

$$\text{I.C.} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad -\infty \le x < \infty$$

by breaking it into steps.

STEP 1 [*Replacing* (x,t) *by new canonical coordinates* (ξ,μ)]

To solve (15), it turns out that if we replace the two independent variables x and t by the two new space-time coordinates (ξ,μ).

$$\xi = x + ct$$

$$\mu = x - ct$$

The PDE $u_{tt} = c^2 u_{xx}$ is transformed into $u_{\mu\xi} = 0$.

This is really very easy to see, since a simple application of the chain rule gives

$$u_x = u_\xi + u_\mu$$

$$u_t = c(u_\xi - u_\mu)$$

$$u_{xx} = u_{\xi\xi} + u_{\mu\xi} + u_{\mu\mu}$$

$$u_{tt} = c^2(u_{\xi\xi} - u_{\mu\xi} + u_{\mu\mu})$$

and, hence, substituting the expressions for u_{tt} and u_{xx} into the wave equation gives

$$c^2(u_{\xi\xi} + u_{\mu\xi} + u_{\mu\mu}) = c^2(u_{\xi\xi} - u_{\mu\xi} + u_{\mu\mu})$$

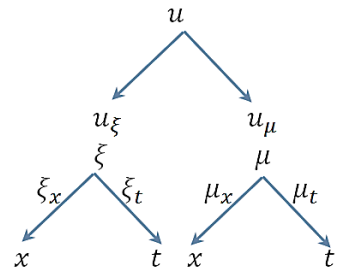
$$4u_{\mu\xi} = 0 \Rightarrow u_{\mu\xi} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \mu}\cdot\frac{\partial \mu}{\partial x}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \mu}\cdot\frac{\partial \mu}{\partial t}$$

$$\frac{\partial u}{\partial \xi} = u_\xi, \frac{\partial u}{\partial \mu} = u_\mu$$

$$\frac{\partial \xi}{\partial x} = \frac{\partial \mu}{\partial x} = 1, \frac{\partial \xi}{\partial t} = c, \frac{\partial \mu}{\partial t} = -c$$



This completes step 1.

STEP 2 (*Solving the transformed equations*)

The idea now is that this new equation can be solved easily by two straightforward integrations (first with respect to ξ and then with respect to μ). Integration with respect to ξ gives

$u_\mu(\xi,\mu) = \phi(\mu)$ (an arbitrary function of μ)

and, secondly, integration with respect to μ gives

$$u(\xi,\mu) = \Phi(\mu) + \psi(\xi)$$

where $\Phi(\mu)$ is the antiderivative of $\phi(\mu)$, and $\psi(\xi)$ is any function of ξ. In other words, we can say that the general solution (all solutions) of

$u_{\mu\xi} = 0$ is $u(\xi,\mu) = \phi(\mu) + \psi(\xi)$

where $\phi(\mu)$, $\psi(\xi)$ are arbitrary functions of μ and ξ, respectively.

This completes step 2.

STEP 3. (*transforming back to the original coordinates x and t*)

To find the *general solution* (all solutions) to $u_{tt} = c^2 u_{xx}$ we substitute

$$\xi = x + ct$$
$$\mu = x - ct$$

into

$$u(\xi, \mu) = \phi(\mu) + \psi(\xi)$$

to get

$$u(x,t) = \phi(x - ct) + \psi(x + ct)$$

This is the general solution of the wave equation, and it is interesting in that it physically represents the sum of *any* two *moving waves*, each moving in opposite directions with velocity *c*.

STEP 4 (*Substituting the general solution into the ICs*)

If the reader remembers ODE theory, the general strategy for solving initial-value problems was to find the general solution and then substitute this equation into the ICs in order to find the arbitrary constants. In our current problem, we have an analogous situation; in order to solve our initial-value problem, we first found the general solution

$$u(x,t) = \phi(x - ct) + \psi(x + ct)$$

to the PDE (which contains *two arbitrary functions)* and now substitute this expression into the two ICs

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

to find the arbitrary functions ϕ, ψ. Doing this, we get

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ -c\phi'(x) + c\psi'(x) &= g(x) \end{aligned} \tag{16}$$

integrate the second equation of (16) to get a new expression in $\phi(x)$ and $\psi(x)$ [and then algebraically solve for $\phi(x)$ and $\psi(x)$ from the two equations].

Carrying out this integration on the second equation of (16) by integrating from x_0 to *x*, we get

$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\xi)d\xi + K \tag{17}$$

and so if we algebraically solve for $\phi(x)$ and $\psi(x)$ from the first equation of (16) and (17), we have

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)d\xi$$
$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)d\xi$$

and, hence, the solution to our problem (15) is

$$u(x,t) = \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$$

This is what we were aiming for, and it is called the **D' Alembert solution** to (15).

Examples of the D'Alembert Solution

1. Solve

   P.D.E. $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty \quad 0 < t < \infty$

   I.C. $\begin{cases} u(x,0) = sin(x) \\ u_t(x,0) = 0 \end{cases}$ $\quad -\infty \le x < \infty$

   Solution:

$$u(x,t) = \frac{1}{2}\left[sin(x-ct)+sin(x+ct)\right]$$

2. Solve

   P.D.E. $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty \quad 0 < t < \infty$

   I.C. $\begin{cases} u(x,0) = 0 \\ u_t(x,0) = sin(x) \end{cases}$ $\quad -\infty \le x < \infty$

   Solution:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} sin(\xi)d\xi = \frac{1}{2c}\left[cos(x+ct)-cos(x-ct)\right]$$

PROBLEMS

1. What is the solution to the initial-value problem

   P.D.E. $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty \quad 0 < t < \infty$

   I.C. $\begin{cases} u(x,0) = e^{-x^2} \\ u_t(x,0) = 0 \end{cases}$ $\quad -\infty \le x < \infty$

2. What is the solution to the initial-value problem

   P.D.E. $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty \quad 0 < t < \infty$

   I.C. $\begin{cases} u(x,0) = 0 \\ u_t(x,0) = xe^{-x^2} \end{cases}$ $\quad -\infty \le x < \infty$

***LESSON* 12: The Finite Vibrating String (Standing Waves)**

We show how transverse vibrations of a finite string described by the IBVP

$$\text{P.D.E.} \quad u_{tt} = c^2 u_{xx} \qquad ,0 < x < L \quad , \ 0 < t < \infty \qquad (18)$$

$$\text{B.cs.} \quad \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \qquad ,0 < t < \infty$$

$$\text{I.C.} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad ,0 \le x \le L$$

can be found by the standard technique of separation of variables and to show how the solution $u(x,t)$ can be interpreted of simple vibrations where the shape $X_n(x)$ of these fundamental vibrations are solutions (eigenfunctions) of a certain Sturm-Liouville boundary-value problem.

Separation of Variables Solution to the Finite Vibrating String

To solve the IBVP

$$\text{P.D.E.} \quad u_{tt} = \alpha^2 u_{xx} \qquad ,0 < x < L \quad , \ 0 < t < \infty$$

$$\text{B.cs.} \quad \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \qquad ,0 < t < \infty$$

$$\text{I.C.} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad ,0 \le x \le L$$

that is, solutions of the form

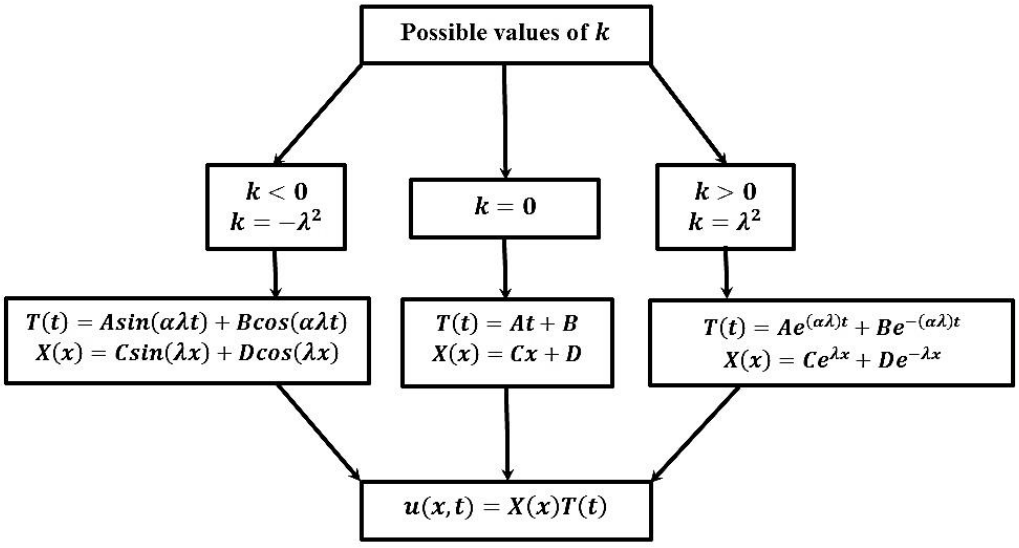
$$u(x,t) = X(x)T(t)$$

Substituting this expression into the wave equation and separating variables gives us the two ODEs

$$T'' - \alpha^2 kT = 0 \qquad T'' + \alpha^2 \lambda^2 T = 0$$
$$X'' - kT = 0 \qquad X'' + \lambda^2 X = 0$$

where the constant *k* can now be any number $-\infty < k < \infty$.

Investigating the solutions of these two ODEs for all different values of *k* yields the diagram



The idea now is to *prune away* all those standing waves that either are unbounded as $t \to \infty$ or else yield only the zero solution when substituted into the BCs $u(0,t) = u(L,t) = 0$. It will be left only negative values of X. give feasible (nonzero and bounded) solutions.

Hence, our goal is to find the constants *A*, *B*, C, and *D* and the negative separation constant A so that the expression

$$u(x,t) = X(x)T(t) = [Csin(\lambda x) + Dcos(\lambda x)] \cdot [Asin(\alpha\lambda t) + Bcos(\alpha\lambda t)] \quad (18)$$

Satisfies the BCs (the sum will still satisfy the PDE and the BC).

Substituting (18) into $u(0,t) = u(L,t) = 0$ gives

$$u(0,t) = X(0)T(t) = D[Asin(\alpha\lambda t) + Bcos(\alpha\lambda t)] = 0$$

$$\Longrightarrow D = 0$$

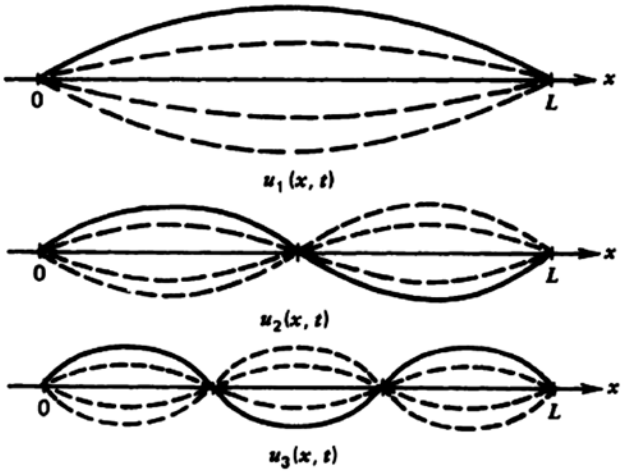
$$u(x,t) = X(L)T(t) = Csin(\lambda L)[Asin(\alpha\lambda t) + Bcos(\alpha\lambda t)] = 0$$

$$\Longrightarrow sin(\lambda L) = 0$$

In other words, the separation constant λ must satisfy $\sin(\lambda L) = 0$ or $\lambda_n = \frac{n\pi}{L}$

Hence,

$$u_n(x,t) = X_n(L)T_n(t) = sin\left(\frac{n\pi x}{L}\right)\left[A_n sin\left(\frac{n\pi\alpha t}{L}\right) + B_n cos\left(\frac{n\pi\alpha t}{L}\right)\right], n = 1,2,3, \ldots$$



Since any sum of these vibrations is also a solution to the PDE and BCs (since the PDE and BCs are linear and homogeneous), we add them together in such a way that the resulting sum *also* agrees with the ICs. This will then be the solution to our problem. Substituting the sum

$$u(x,t) = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left[A_n sin\left(\frac{n\pi\alpha t}{L}\right) + B_n cos\left(\frac{n\pi\alpha t}{L}\right)\right]$$

into the ICs

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

gives the two equations

$$\sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\sum_{n=1}^{\infty} A_n(\frac{n\pi x}{L}) sin\left(\frac{n\pi x}{L}\right) = g(x)$$

and using the orthogonality condition

$$\int_0^L sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

we can find the coefficients A_n and B_n

$$A_n = \frac{2}{n\pi x}\int_0^L g(x)\, sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right) dx$$

So we're done, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left[A_n sin\left(\frac{n\pi\alpha t}{L}\right) + B_n cos\left(\frac{n\pi\alpha t}{L}\right)\right]$$

This completes the problem.

NOTES:

If the initial *velocity* of the string is zero $(g(x) = 0)$, then the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi\alpha t}{L}\right)$$

into simple sine components

$$f(x) = \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right)$$

Example: Solve the IBVP

P.D.E. $u_{tt} = \alpha^2 u_{xx}$ $,0 < x < L \quad , \ 0 < t < \infty$

B.cs. $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ $,0 < t < \infty$

I.C. $\begin{cases} u(x,0) = sin\left(\frac{\pi x}{L}\right) + \frac{1}{2} sin\left(\frac{3\pi x}{L}\right) + \frac{1}{4} sin\left(\frac{5\pi x}{L}\right) \\ u_t(x,0) = 0 \end{cases}$ $, 0 \leq x \leq L$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi\alpha t}{L}\right)$$

Since $B_1 = 1, B_2 = 0, B_3 = \frac{1}{2}, B_4 = 0, B_5 = \frac{1}{4}, B_n = 0, for\ n \geq 6.$

Boundary Conditions Associated with the Wave Equation

Some of the various types of BCs that are associated with physical problems of this kind. Here, we will stick to *one dimensional problems* where the BCs (linear ones) are generally grouped into one of three kinds:

1. *Controlled end points* (first kind)
$$u(0,t) = g_1(t)$$
$$u(L,t) = g_2(t)$$
2. *Force given on the boundaries* (second kind)
$$u_x(0,t) = g_1(t)$$
$$u_x(L,t) = g_2(t)$$
3. *Elastic attachment on the boundaries* (third kind)
$$u_x(0,t) - \gamma_1 u(0,t) = g_1(t)$$
$$u_x(L,t) - \gamma_2 u(L,t) = g_2(t)$$

PROBLEMS

1. Find the solution to the vibrating-string problem

P.D.E. $u_{tt} = \alpha^2 u_{xx}$ $,0 < x < L \quad , \ 0 < t < \infty$

B.cs. $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ $,0 < t < \infty$

I.C. $\begin{cases} u(x,0) = sin\left(\frac{\pi x}{L}\right) + \frac{1}{2} sin\left(\frac{3\pi x}{L}\right) \\ u_t(x,0) = 0 \end{cases}$ $, 0 \le x \le L$

2. What is the solution of the vibrating-string problem

P.D.E. $u_{tt} = \alpha^2 u_{xx}$ $,0 < x < L \quad , \ 0 < t < \infty$

B.cs. $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ $,0 < t < \infty$

I.C. $\begin{cases} u(x,0) = sin\left(\frac{3\pi x}{L}\right) \\ u_t(x,0) = \left(\frac{3\pi \alpha}{L}\right) sin\left(\frac{3\pi x}{L}\right) \end{cases}$ $, 0 \le x \le L$

3. What is the solution of the vibrating-string problem if the ICs are

P.D.E. $u_{tt} = \alpha^2 u_{xx}$ $,0 < x < L \quad , \ 0 < t < \infty$

B.cs. $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ $,0 < t < \infty$

I.C. $\begin{cases} u(x,0) = sin\left(\frac{\pi x}{L}\right) + \frac{1}{2} sin\left(\frac{3\pi x}{L}\right) \\ u_t(x,0) = 0 \end{cases}$ $, 0 \le x \le L$

***LESSON* 13: First-Order Equations (Method of Characteristics)**

We shall introduce the notion of first-order partial differential equations and introduce an important technique for solving initial-value problems (method of characteristics).

The problem we will solve is the initial-value problem

P.D.E. $a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

I.C. $u(x,0) = \phi(x) \quad , -\infty < x < \infty$

Note that this is the first time we've solved problems with variable coefficients.

It turns out that if we change coordinates from (x,t) to appropriate new coordinates (s,τ) (characteristic coordinates), then our differential equation becomes an *ordinary differential equation.*

Hence, we solve the ODE to find $u(s,\tau)$, and then the last step is to plug in the values of s and τ in terms of x and t to get $u(x,t)$.

Recall that we solved the diffusion equation

$$u_t = \alpha^2 u_{xx} - v u_x \quad , -\infty < x < \infty, 0 < t < \infty$$

the constant α^2 stood for the amount of diffusion, while v stood for the velocity of the medium. Hence, if $\alpha = 0$ (no diffusion), it's clear (since we only have convection) that the solution will travel along the x-axis with velocity v.

In other words, if the initial solution is $u(x,0) = \phi(x)$ then the solution to

$$u_t = -v u_x$$

would be $u(x,t) = \phi(x - vt)$.

Let us back to our basic problem

P.D.E. $a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

I.C. $u(x,0) = \phi(x) \quad , -\infty < x < \infty$

The idea is to introduce *two new coordinates* s and τ (to replace x and t) that have the properties

s will change along the *characteristic* curves.

τ will change along the *initial* curve (most likely the line $t = 0$).

First, consider the new coordinate s. By choosing s having the above property, the PDE

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0$$

is transformed into the ODE

$$\frac{du}{ds} + c(x,t)u = 0$$

we pick the *characteristic curves* $\{[x(s), t(s)]: 0 < s < \infty\}$ so that

$$\frac{dx}{ds} = a(x,t), \qquad \frac{dt}{ds} = b(x,t)$$

By doing this, it is clear that

$$\frac{du}{ds} = \frac{dx}{ds}u_x + \frac{dt}{ds}u_t = a(x,t)u_x + b(x,t)u_t$$

In other words, along the curves $\{[x(s), t(s)]: 0 < s < \infty\}$, the PDE becomes an ODE.

Example:

Suppose we want to solve the following IVP with constant coefficients:

P.D.E. $u_x + u_t + 2u = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

I.C. $u(x,0) = \sin x \quad , -\infty < x < \infty$

STEP 1 Find the characteristics (along which the initial data propagate)

$$\frac{dx}{ds} = 1, \qquad \frac{dt}{ds} = 1$$

Solving these equations gives (remembers is now the independent variable)

$$x(s) = s + c_1, \qquad t(s) = s + c_2$$

To evaluate the constants c_1 and c_2, *let* $s = 0$,

$$x(0) = \tau, \qquad t(0) = 0$$

Doing this gives values of $c_1 = \tau$ and $c_2 = 0$. Hence, our characteristic curves are

$$x = s + \tau, \qquad t = s$$

STEP 2 Using this new coordinate system, we change the PDE to the ODE

O.D.E. $\frac{du}{ds} + 2u = 0$

I.C. $u(0) = \sin \tau$

Solving this IVP, we get

$$u(s,\tau) = \sin \tau\, e^{-2s}$$

STEP 3 We now solve the transformation

$$x = s + \tau, \qquad t = s$$

for s and τ in terms of x and t. We have

$$s = t, \qquad \tau = t - x$$

Hence,

$$u(x,t) = \sin(t - x)\, e^{-2t}$$

General Strategy for Solving the First-Order Equation

Suppose we are given the IVP

P.D.E. $a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

I.C. $u(x,0) = f(x) \quad , -\infty < x < \infty$

STEP 1 Solve the two ODEs (characteristic equations)

$$\frac{dx}{ds} = a(x,t), \qquad \frac{dt}{ds} = b(x,t)$$

Find the constants of integration by setting $x(0) = \tau$ and $t(0) = 0$. We now have the transformation from (x, t) into (s, τ)

$$x = x(s, \tau)$$
$$t = t(s, \tau)$$

STEP 2 Solve the ODE

ODE $\frac{du}{ds} + c\big(x(s,\tau), t(s,\tau)\big)u = 0$

I.C. $u(0) = f(\tau)$

STEP 3 After solving the ODE (with the ICs), we get a solution $u(s, \tau)$. We must now solve for s and τ in terms of x and t (from the transformation we found in Step 1) and substitute these values into $u(s, \tau)$.

We now apply these steps in the following example (note variable coefficients).

Example

P.D.E. $xu_x + u_t + tu = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

I.C. $u(x, 0) = F(x)$ (an *arbitrary* initial wave)

STEP 1

$\frac{dx}{ds} = x$ has solution $x(s) = c_1 e^s$

$\frac{dt}{ds} = 1$ has solution $t(s) = s + c_2$

Letting $x(0) = \tau$ and $t(0) = 0$, gives $c_1 = \tau$ and $c_2 = 0$.

Hence, the transformation into new coordinates is

$x(s) = \tau e^s$

$t(s) = s$

STEP 2

Solve

O.D.E. $\frac{du}{ds} + su = 0, \qquad 0 < s < \infty$

I.C. $u(0) = F(\tau)$

The solution is

$$u(s, \tau) = F(\tau) e^{-s^2/2}$$

STEP 3

Solving for s and τ gives $s = t$, $\tau = xe^{-t}$. Hence, the solution

$$u(x, t) = F(xe^{-t}) e^{-t^2/2}$$

In other words, if the IC were $u(x, 0) = \sin x$, then the solution would be

$$u(x, t) = \sin(xe^{-t}) e^{-t^2/2}$$

PROBLEMS

1. Solve the simple concentration problem

   P.D.E. $xu_x + u_t = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

   I.C. $u(x,0) = cos\, x \quad , -\infty < x < \infty$

   By Method of Characteristics.

2. Solve the problem

   P.D.E. $xu_x + tu_t + 2u = 0 \quad , -\infty < x < \infty,\ 1 < t < \infty$

   I.C. $u(x,1) = sin\, x \quad , -\infty < x < \infty$

   By Method of Characteristics. (Note that time starts at $t = 1$ here).

3. Solve

   P.D.E. $u_x + u_t + tu = 0 \quad , -\infty < x < \infty,\ 0 < t < \infty$

   I.C. $u(x,0) = F(x) \quad , -\infty < x < \infty$

   By Method of Characteristics.

***LESSON* 14: Systems of PDEs**

To point out that many physical systems (particularly in fluid dynamics) cannot be described by a single PDE but, in fact, are modeled by a system of interlocking equations.

In these equations, several unknown functions like $pressure = p(x, y, z, t)$, $density = p(x, y, z, t)$ and $temperature = T(x, y, z, t)$ (and their partial derivatives) are described by physical laws, and we seek to find all of these functions simultaneously.

It will also be shown how the *linear* system of equations

$$u_t + Au_x = 0$$

can be solved by decomposing it into a new system of independent equations

$$v_t + Bv_x = 0$$

(This way, each equation can be solved separately.)

In many areas of science, several unknown quantities (and their derivatives) are related by more than one equation. For example, in fluid dynamics, the four equations

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = 0$$

$$v_t + uv_x + vv_y + \frac{1}{\rho}p_y = 0$$

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$

$$\left(\frac{\rho}{\rho^\gamma}\right)_t + u\left(\frac{p}{\rho^\gamma}\right)_x + v\left(\frac{p}{\rho^\gamma}\right)_y = 0$$

where

$p(x, y, t) = Pressure\ of\ a\ fluid$

$u(x, y, t) = x - velocity\ of\ a\ fluid$

$v(x, y, t) = y - velocity\ of\ a\ fluid$

$p(x, y, t) = Density\ of\ a\ fluid$

constitute a *nonlinear* system of PDEs known as **Euler's equations of motion for a fluid.**

Diagonalization of a Matrix

If **A** is an *n* by *n* matrix with *n* distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and if

$$\boldsymbol{P} = \left[\boldsymbol{X_1} \middle| \boldsymbol{X_2} \middle| \cdots \middle| \boldsymbol{X_n}\right] = \left[\begin{array}{c|c|c|c} \boldsymbol{x_{11}} & \boldsymbol{x_{12}} & \cdots & \boldsymbol{x_{1n}} \\ \boldsymbol{x_{21}} & \boldsymbol{x_{22}} & \cdots & \boldsymbol{x_{2n}} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{x_{n1}} & \boldsymbol{x_{n2}} & \cdots & \boldsymbol{x_{nn}} \end{array}\right]$$

is the matrix whose *n-th column* is the eigenvector $\boldsymbol{X_n}$ corresponding to λ_n then

$$\boldsymbol{P^{-1}AP = B}$$

where $\boldsymbol{P^{-1}}$ is the inverse of $\boldsymbol{P}$, and **B** is the diagonal matrix

$$\boldsymbol{B} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

For example, the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -4$, with corresponding eigenvectors

$$\boldsymbol{X_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{X_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence, the following product:

$$\boldsymbol{P^{-1}AP} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

is equal to the diagonal matrix

$$\boldsymbol{B} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

We will now solve a simple system of two equations (along with their ICs).

Solution of the Linear System $\boldsymbol{u_t + Au_x = 0}$

Consider the intitial-value problem consisting of two PDEs and two ICs

PDE 1 $\frac{\partial u_1}{\partial t} + 8\frac{\partial u_2}{\partial x} = 0 \qquad -\infty < x < \infty,\ 0 < t < \infty$

PDE 2 $\frac{\partial u_2}{\partial t} + 2\frac{\partial u_1}{\partial x} = 0$

I.C. 1 $u_1(x,0) = f(x) \qquad -\infty < x < \infty$

I.C. 2 $u_2(x,0) = g(x)$

This problem might correspond to finding $u_1(x,t) = pressure$ and $u_2(x,t) = density$ as a function of (a space variable) x and (time) t when we are given the initial pressure and density.

We start by writing the two PDEs in matrix form

$$\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$u_t + Au_x = 0$$

where

$$A = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix},\ u_t = \begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{bmatrix},\ u_x = \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \end{bmatrix} 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, we introduce a *new* unknown (change of variables technique)

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

by means of the *transformation*:

$$u = Pv$$

where P is the matrix whose columns are the eigenvectors of A. It turns out that v will satisfy a very easy system of equations (the two equations in the new unknowns v_1 and v_2 are *independent* of each other), and hence, we can find v_1 and v_2 easily. After we find v_1 and v_2, we then compute u_1 and u_2 from

$$u = Pv$$

we can differentiate these equations to get

$$\frac{\partial u}{\partial t} = P\frac{\partial v}{\partial t}$$
$$\frac{\partial u}{\partial x} = P\frac{\partial v}{\partial x}$$

now substitute equations into our system

$$u_t + Au_x = 0$$

to get

$$Pv_t + APv_x = 0$$

and if we multiply each side of this vector equation by P^{-1}, we have

$$v_t + P^{-1}APv_x = 0$$

or

$$v_t + Bv_x = 0$$

Since we have already seen that

$$\boldsymbol{B} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

for our matrix B, we can get the two equations

$$\frac{\partial v_1}{\partial t} + 4\frac{\partial v_1}{\partial x} = 0$$
$$\frac{\partial v_2}{\partial t} - 4\frac{\partial v_2}{\partial x} = 0$$

These are the two *uncoupled* equations that we can solve independently.

$$v_1(x,t) = \phi(x - 4t)$$
$$v_2(x,t) = \psi(x + 4t)$$

(where ϕ and ψ are arbitrary differentiable functions).

These are the values of v_1 and v_2 To obtain our general solution u, we compute

$$u = Pv = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \phi(x - 4t) \\ \psi(x + 4t) \end{bmatrix} = \begin{bmatrix} 2\phi(x - 4t) - 2\psi(x + 4t) \\ \phi(x - 4t) - \psi(x + 4t) \end{bmatrix}$$

In other words,

$$u_1(x,t) = 2\phi(x-4t) - 2\psi(x+4t)$$
$$u_2(x,t) = \phi(x-4t) + \psi(x+4t)$$

Hence, we have found the general solution to the two PDEs of problem.

A typical solution (there are an infinite number) would be to let

$$\phi(\xi) = sin\,\xi$$
$$\psi(\xi) = \xi^2$$

and, hence

$$u_1(x,t) = 2sin(x-4t) - 2(x+4t)^2$$
$$u_2(x,t) = sin(x-4t) + (x+4t)^2$$

We now substitute general solution into the ICs

$$u_1(x,0) = f(x)$$
$$u_2(x,0) = g(x)$$

to get

$$2\phi(x) - 2\psi(x) = f(x)$$
$$\phi(x) + \psi(x) = g(x)$$

Solving for ϕ and ψ gives

$$\phi(x) = \frac{1}{4}[f(x) + 2g(x)]$$
$$\psi(x) = \frac{1}{4}[2g(x) - f(x)]$$

hence, the solution to IVP

$$\begin{aligned} u_1(x,t) &= 2\phi(x-4t) - 2\psi(x+4t) \\ &= \frac{1}{2}[f(x-4t) + 2g(x-4t)] - \frac{1}{2}[2g(x+4t) - f(x+4t)] \\ u_2(x,t) &= \phi(x-4t) + \psi(x+4t) \\ &= \frac{1}{4}[f(x-4t) + 2g(x-4t)] + \frac{1}{4}[2g(x+4t) - f(x+4t)] \end{aligned}$$

PROBLEMS

1. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

2. Using the results of problem 1, what is the general solution of the system

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0$$
$$\frac{\partial u_2}{\partial t} + 4\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0$$

HINT First rewrite the system in matrix form $v_t + Bv_x = 0$

LESSON 15: Elliptic-Type Problems

The Laplacian (an Intuitive Description)

The Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is probably the most important operator in mathematical physics.

Some Basic Lows of Physics

- The heat equation $u_t = \alpha^2 \nabla^2 u$.
- The wave equation $u_{tt} = \alpha^2 \nabla^2 u$.
- Laplace's equation $\nabla^2 u = 0$.
- Poisson's equation $\nabla^2 u = f$.

The Intuitive Meaning of the Laplacian

$$\nabla^2 u = u_{xx} + u_{yy} \qquad \textbf{(Two dimensions)}$$

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} \qquad \textbf{(Three dimensions)}$$

However, in many problems, it is necessary to write this equation in other coordinates. For example, if the boundary of the region is a circle, then we must use *polar coordinates* (r, θ), while if we are in three dimensions and our boundary is a sphere, then we ·1urn to *spherical coordinates* (r, θ, ϕ).

The Five Major Coordinate Systems in Two and Three Dimensions

1. Cartesian system in two dimensions.
2. Cartesian system in three dimensions.
3. Polar coordinates (two dimensions).
4. Cylindrical coordinates (three dimensions).
5. Spherical coordinates (three dimensions).

Polar coordinates are defined by

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$

$$\theta = \tan^{-1}\frac{y}{x} \qquad y = r\sin\theta$$

Cylindrical coordinates are defined by

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$

$$\theta = \tan^{-1}\frac{y}{x} \qquad y = r\sin\theta$$

$$z = z \qquad z = z$$

Spherical coordinates are defined by

$$r^2 = x^2 + y^2 + z^2 \qquad x = r\sin\phi\cos\theta$$
$$\tan\theta = \frac{y}{x} \qquad y = r\sin\phi\sin\theta$$
$$\cos\phi = \frac{z}{r} \qquad z = r\cos\phi$$

Changing Coordinates

We are ready to change coordinates of the two-dimensional Laplacian is transformed into *polar* coordinates.

$$\nabla^2 u = u_{xx} + u_{yy}$$

To find the new Laplacian, using the chain rule:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$
$$= u_r \cdot r_x + u_\theta \cdot \theta_x$$
$$= u_r \cdot \cos\theta + u_\theta \cdot \frac{\sin\theta}{r}$$

By a similar argument, we have

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$
$$= u_r \cdot r_y + u_\theta \cdot \theta_y$$
$$= u_r \cdot \sin\theta + u_\theta \cdot \frac{\cos\theta}{r}$$

If u depends on r and θ (which, in turn, depends on x and y), then u_x will also depend) on the same variables. So, we have

$$u_{xx} = (u_x)_x = \frac{\partial u_x}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u_x}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$
$$= \frac{\partial}{\partial r}\left(u_r \cdot \cos\theta + u_\theta \cdot \frac{\sin\theta}{r}\right)\cdot r_x + \frac{\partial}{\partial \theta}\left(u_r \cdot \cos\theta + u_\theta \cdot \frac{\sin\theta}{r}\right)\cdot \theta_x$$
$$= \left(u_{rr}\cdot\cos\theta + u_\theta \cdot \frac{\sin\theta}{r^2} - u_{r\theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$
$$+ \left(u_{r\theta}\cdot\cos\theta - u_r\cdot\sin\theta - u_\theta\cdot\frac{\cos\theta}{r} - u_{\theta\theta}\cdot\frac{\sin\theta}{r}\right)\cdot -\frac{\sin\theta}{r}$$
$$= u_{rr}\cdot\cos^2\theta + u_\theta\cdot\frac{\sin\theta\cos\theta}{r^2} - u_{\theta r}\frac{\sin\theta\cos\theta}{r}$$
$$-u_{r\theta}\cdot\frac{\sin\theta\cos\theta}{r} + u_r\cdot\frac{\sin^2\theta}{r} + u_\theta\cdot\frac{\sin\theta\cos\theta}{r^2} + u_{\theta\theta}\cdot\frac{\sin^2\theta}{r^2}$$
$$= u_{rr}\cdot\cos^2\theta + u_r\cdot\frac{\sin^2\theta}{r} + 2u_\theta\cdot\frac{\sin\theta\cos\theta}{r^2} - 2u_{\theta r}\frac{\sin\theta\cos\theta}{r} + u_{\theta\theta}\cdot\frac{\sin^2\theta}{r^2}$$

$$\frac{d}{dx}r = \frac{d}{dx}\sqrt{x^2+y^2}$$
$$= \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$
$$\frac{d}{dx}\theta = \frac{d}{dx}tan^{-1}\frac{y}{x} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$
$$= \frac{-y}{x^2}\cdot\frac{x^2}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$
$$\frac{d}{dy}r = \frac{d}{dy}\sqrt{x^2+y^2}$$
$$= \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$
$$\frac{d}{dy}\theta = \frac{d}{dy}tan^{-1}\frac{y}{x} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}$$
$$= \frac{1}{x}\cdot\frac{x^2}{x^2+y^2} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2}$$
$$= \frac{\cos\theta}{r}$$

By a similar argument, we get

$$u_{yy} = (u_y)_y = \frac{\partial u_y}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u_y}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \left(u_{rr}\cdot sin\,\theta - u_\theta\cdot\frac{cos\,\theta}{r^2} + u_{r\theta}\frac{cos\,\theta}{r}\right)\cdot sin\,\theta$$

$$+\left(u_{r\theta}\cdot sin\,\theta + u_r\cdot cos\,\theta - u_\theta\cdot\frac{sin\,\theta}{r} + u_{\theta\theta}\cdot\frac{cos\,\theta}{r}\right)\cdot\frac{cos\,\theta}{r}$$

$$= u_{rr}\cdot sin^2\,\theta - 2u_\theta\cdot\frac{sin\,\theta\,cos\,\theta}{r^2} + 2u_{r\theta}\frac{sin\,\theta\,cos\,\theta}{r} + + u_r\cdot\frac{cos^2\,\theta}{r} + u_{\theta\theta}\cdot\frac{cos^2\,\theta}{r^2}$$

Adding u_{xx} and u_{yy} we get the desired result

$$\nabla^2 u = u_{xx} + u_{yy} =$$

$$= u_{rr}\cdot cos^2\,\theta + u_r\cdot\frac{sin^2\,\theta}{r} + 2u_\theta\cdot\frac{sin\,\theta\,cos\,\theta}{r^2} - 2u_{\theta r}\frac{sin\,\theta\,cos\,\theta}{r} + u_{\theta\theta}\cdot\frac{sin^2\,\theta}{r^2}$$

$$+u_{rr}\cdot sin^2\,\theta + u_r\cdot\frac{cos^2\,\theta}{r} - 2u_\theta\cdot\frac{sin\,\theta\,cos\,\theta}{r^2} + 2u_{r\theta}\frac{sin\,\theta\,cos\,\theta}{r} + u_{\theta\theta}\cdot\frac{cos^2\,\theta}{r^2}$$

$$= u_{rr}\cdot(sin^2\,\theta + cos^2\,\theta) + u_r\cdot\frac{(sin^2\,\theta + cos^2\,\theta)}{r} + u_{\theta\theta}\cdot\frac{(sin^2\,\theta + cos^2\,\theta)}{r^2}$$

$$= u_{rr} + \frac{1}{r}\cdot u_r + \frac{1}{r^2}\cdot u_{\theta\theta}$$

Similar analysis for the three-dimensional Laplacian

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

Changing to *cylindrical coordinates*, we can show

$$\nabla^2 u = u_{rr} + \frac{1}{r}\cdot u_r + \frac{1}{r^2}\cdot u_{\theta\theta} + u_{zz}$$

Finally, if we write the Laplacian in *spherical coordinates*, we have

$$\nabla^2 u = u_{rr} + \frac{2}{r}\cdot u_r + \frac{1}{r^2}\cdot u_{\phi\phi} + \frac{cot\,\theta}{r^2}\cdot u_\phi + \frac{1}{r^2\,sin^2\,\phi}\cdot u_{\theta\theta}$$

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